

# FINITE GROUP ACTIONS ON HOMOLOGY SPHERES AND MANIFOLDS WITH NONZERO EULER CHARACTERISTIC

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**ABSTRACT.** Let  $X$  be a smooth manifold belonging to one of these three collections: (1) acyclic manifolds (compact or not, possibly with boundary), (2) compact manifolds (possibly with boundary) with nonzero Euler characteristic, and (3) homology spheres. We prove the existence of a constant  $C$  such that any finite group acting effectively and smoothly on  $X$  has an abelian subgroup of index at most  $C$ . The proof uses a result on finite groups by Alexandre Turull and the author which is based on the classification of finite simple groups. If  $X$  is compact and its cohomology is torsion free and supported in even degrees, we also prove the existence of a constant  $C'$  such that any finite abelian group  $A$  acting on  $X$  has a subgroup  $A_0$  of index at most  $C'$  such that  $\chi(X^{A_0}) = \chi(X)$ .

## 1. INTRODUCTION

In this paper we are interested on smooth finite group actions on manifolds. Two general and natural questions in this subject are to characterize which finite groups admit effective actions on a given manifold, and to study geometric properties such as existence of fixed points for general actions. These questions have attracted attention from the first works on finite transformation groups and have been a continuous source of inspiration ever since (see [2, 3, 9] for general introductions). Despite all the progress, a general classification of which groups admit effective actions on a given manifold seems to be completely out of reach at present, except in very particular examples (either low dimensional manifolds, or manifolds which have none or very few finite order automorphisms, see e.g. [7]).

Our aim is to study *weak* versions of the preceding questions. More precisely, given a manifold  $X$ , we want to find properties  $\mathcal{P}$  admitting a constant  $C$  (depending on  $X$  and  $\mathcal{P}$ ) such that any finite group  $G$  acting effectively on  $X$  has some subgroup  $G_0$  satisfying  $\mathcal{P}$  and  $[G : G_0] \leq C$ . The properties  $\mathcal{P}$  which we are going to consider are commutativity and existence of fixed points.

1.1. This is the first main result of this paper.

**Theorem 1.1.** *Let  $X$  be a smooth manifold belonging to one of these three collections: (1) acyclic manifolds (compact or not, possibly with boundary), (2) compact manifolds*

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(possibly with boundary) with nonzero Euler characteristic, and (3) homology spheres. There exist constants  $C, d$  such that any finite group  $G$  acting effectively and smoothly on  $X$  has an abelian subgroup  $A$  which can be generated by  $d$  elements and has index  $[G : A] \leq C$ .

Replacing  $C$  by a bigger constant, we can assume that  $A$  is normal in  $G$  (indeed, for any inclusion of finite groups  $\Gamma' \subseteq \Gamma$  there is a subgroup  $\Gamma_0 \subseteq \Gamma'$  which is normal in  $\Gamma$  and such that  $[\Gamma : \Gamma_0] \leq [\Gamma : \Gamma']!$ ). So if  $X$  is any of the manifolds included in Theorem 1.1 then there is an integer  $\delta$  and a finite collection of finite groups  $G_1, \dots, G_r$  such that any finite group  $G$  acting effectively and smoothly on  $X$  sits in an exact sequence

$$(1) \quad 1 \rightarrow \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_\delta} \rightarrow G \rightarrow G_j \rightarrow 1,$$

where  $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$  and  $\delta \leq d$ . Hence, at least theoretically, it should be possible to classify all finite groups admitting effective actions on such manifolds in terms of finitely many parameters: namely, the group  $G_j$ , the numbers  $n_1, \dots, n_\delta$ , and the extension class of (1); the latter is classified by a  $G_j$ -module structure on  $M := \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_\delta}$  and an element of  $H^2(G_j; M)$  (see e.g. [4, Theorem IV.3.12]).

The proof of Theorem 1.1 uses the classification of finite simple groups through a theorem proved jointly by Alexandre Turull and the author in [31] (see Theorem 3.1 below).

Theorem 1.1 gives a positive partial answer to a conjecture of Étienne Ghys, according to which for *any* compact manifold  $X$  there is a constant  $C$  such that any finite group  $G$  acting smoothly and effectively on  $X$  has an abelian subgroup  $A$  of index at most  $C$  (see Question 13.1 in [12]<sup>1</sup>). The particular case in which  $X$  is a sphere was also independently asked in several talks by Walter Feit<sup>2</sup> and later by Bruno Zimmermann [39, §5]. The bound on the number of generators of the abelian group in the statement of Theorem 1.1 is not part of Ghys's conjecture, and follows immediately from a theorem of Mann and Su [22].

Ghys's conjecture was inspired by the following classic result of Jordan [20].

**Theorem 1.2** (Jordan). *For any  $n \in \mathbb{N}$  there exists a constant  $C_n$  such that if  $G$  is a finite subgroup of  $\mathrm{GL}(n, \mathbb{R})$  then  $G$  has an abelian subgroup  $A \subseteq G$  of index at most  $C_n$ .*

Jordan's theorem has also inspired the following terminology, introduced by Popov [34]: a group  $\Gamma$  is said to be Jordan if any finite subgroup  $G \subseteq \Gamma$  has an abelian subgroup whose index in  $G$  is bounded above by a constant depending only on  $\Gamma$ . Hence, Jordan's theorem can be restated by saying that  $\mathrm{GL}(n, \mathbb{C})$  is Jordan for every  $n$ , and Ghys's conjecture states that the diffeomorphism group of any compact manifold is Jordan.

The question of whether automorphism groups of a given geometric structure is Jordan has been asked in other contexts besides smooth manifolds. Serre asked whether high

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<sup>1</sup>This conjecture was discussed in several talks by Ghys [14] (I thank É. Ghys for this information), but apparently it was in [12] when it appeared in print for the first time.

<sup>2</sup>I thank I. Hambleton for this information.

dimensional Cremona groups are Jordan [38, Question 6.1], and Popov extended the question to birational groups and automorphism groups of general algebraic varieties (see [36] for a survey). Note that Theorem 1.1 implies in particular that the group of automorphisms of  $\mathbb{C}^n$  as an affine algebraic manifold is Jordan. The question is also interesting in symplectic geometry (see [30] and below).

Other partial cases of Ghys's conjecture were previously known to be true: it is an easy exercise to prove it for compact manifolds of dimension at most 2 (see [25, Theorem 1.3] for the case of surfaces); for compact 3-manifolds it was proved by Zimmermann in [40]; for compact 4-manifolds with nonzero Euler characteristic it was proved by the author in [27]; finally, the author proved in [25] Ghys's conjecture for closed manifolds admitting a nonzero top dimensional cohomology class expressible as a product of one dimensional classes (e.g. tori). For open manifolds the analogous statement of Ghys conjecture was known to be true in some cases. For  $\mathbb{R}$  and  $\mathbb{R}^2$  it is an easy exercise; for  $\mathbb{R}^3$  it follows from the work of Meeks and Yau on minimal surfaces, see Theorem 4 in [23]; Guazzi, Mecchia and Zimmermann proved in [15] that the diffeomorphism groups of  $\mathbb{R}^3$  and  $\mathbb{R}^4$  are Jordan using much more elementary methods than those of [23]; finally, Zimmermann proved in [40] that the diffeomorphism groups of  $\mathbb{R}^5$  and  $\mathbb{R}^6$  are Jordan using [31].

An example of connected open 4-manifold whose diffeomorphism group is not Jordan was given by Popov in [35].

Recently, Csikós, Pyber, and Szabó [8] found a counterexample to Ghys's conjecture, by showing that the diffeomorphism group of  $T^2 \times S^2$  is not Jordan (in contrast, the symplectomorphism group of any symplectic form on  $T^2 \times S^2$  is Jordan, see [30]). Many other counterexamples to Ghys's conjecture can be obtained using the ideas in [8]: in particular, for any manifold  $M$  supporting an effective action of  $SU(2)$  or  $SO(3, \mathbb{R})$  the diffeomorphism group of  $T^2 \times M$  is not Jordan (see [29]). In view of this, it is an intriguing question to characterize for which compact smooth manifolds Ghys's conjecture is true.

1.2. Our second main result concerns existence of fixed points. We say that a compact manifold  $X$  has no odd cohomology if its integral cohomology is torsion free and supported in even degrees. This implies that the Euler characteristic of any connected component of  $X$  is nonzero. Note that if  $X$  is orientable and closed then the assumption that  $H^*(X; \mathbb{Z})$  is supported in even degrees implies, by Poincaré duality and the universal coefficient theorem, that the cohomology is torsion free.

**Theorem 1.3.** *Let  $X$  be a compact smooth manifold (possibly with boundary) without odd cohomology. There exists a constant  $C$  such that any finite abelian group  $A$  acting smoothly on  $X$  contains a subgroup  $A_0$  satisfying  $[A : A_0] \leq C$  and  $\chi(Y^{A_0}) = \chi(Y)$  for every connected component  $Y \subseteq X$ .*

Theorem 1.3 implies that  $A_0$  preserves the connected components of  $X$  and for any connected component  $Y \subseteq X$  the fixed point set  $Y^{A_0}$  is nonempty. We remark that Theorem 1.3 is not a formal consequence of the particular case in which  $X$  is connected.

To prove Theorem 1.3 we introduce a condition on finite abelian smooth group actions on the manifold  $X$  called  $\lambda$ -stability, where  $\lambda$  is an integer. We prove that if  $\lambda$  is big enough (depending on  $X$ ) then for any  $\lambda$ -stable smooth action of a finite abelian group  $\Gamma$  on  $X$  there exists some  $\gamma \in \Gamma$  satisfying  $X^\Gamma = X^\gamma$ ; then the Euler characteristic of  $X^\gamma$  can be computed using Lefschetz' formula [9, Exercise 6.17.3]. Furthermore, for any  $\lambda$  there exists a constant  $C_\lambda$  (depending on  $\lambda$  and  $X$ ) such that for any abelian group  $\Gamma$  acting smoothly on  $X$  has a subgroup  $\Gamma_0$  whose action on  $X$  is  $\lambda$ -stable and  $[\Gamma : \Gamma_0] \leq C_\lambda$ .

Theorem 1.3 can be combined with Theorem 1.1 to yield the following.

**Corollary 1.4.** *Let  $X$  be a smooth compact manifold (possibly with boundary) without odd cohomology. There exists a constant  $C$  such that any finite group  $G$  acting smoothly on  $X$  has a subgroup  $G_0$  satisfying  $[G : G_0] \leq C$  and  $|X^{G_0}| \geq \chi(X)$ .*

This corollary is nontrivial even for high dimensional disks, since disks of high enough dimension support finite group actions without fixed points. The first example of such actions was found by Floyd and Richardson (see [13] and also [3, Chap. I, §8]), and a complete characterization of finite groups admitting smooth actions on disks without fixed points was given by Oliver (see Theorem 7 of [32]). In particular, Oliver proves that a finite abelian group has a smooth fixed point free action on a disk if and only if it has three or more noncyclic Sylow subgroups. There is a similar story for finite group actions on spheres with a unique fixed point, see [33]. Neither the methods of [32] nor those of [33] give a precise control on the dimension of the disks or spheres supporting the finite group action. Some partial results, restricted either to low dimensions or to actions of some concrete finite groups, on the dimensions of disks (resp. spheres) supporting group actions with none (resp. one) fixed points, are proved in [1, 5].

Compactness is an essential condition in Theorem 1.3. Indeed, the analogous statement for smooth finite group actions on  $\mathbb{R}^n$  is false for  $n \geq 7$ : by a theorem of Haynes, Kwasik, Mast and Schultz [17], if  $n, r$  are natural numbers,  $n \geq 7$  and  $r$  is not a prime power, then there exists a smooth diffeomorphism of  $\mathbb{R}^n$  of order  $r$  without fixed points; taking  $r = pq$  with  $p, q$  different primes, it follows that there is a smooth action of  $G = \mathbb{Z}_r$  on  $\mathbb{R}^n$  such that for every  $x \in \mathbb{R}^n$  the isotropy group  $G_x$  is trivial or isomorphic to  $\mathbb{Z}_p$  or  $\mathbb{Z}_q$ ; in particular  $[G : G_x] \geq \min\{p, q\}$ . Since  $p, q$  can both be chosen to be arbitrarily big, Theorem 1.3 can not be true for actions on  $\mathbb{R}^n$ . (I thank I. Hambleton for this observation.)

**1.3. The constants in Theorems 1.1 and 1.3.** A natural question which we do not address here is to find the optimal values of  $C$  in Theorem 1.1. We also do not estimate the constants that arise from our arguments; doing so would require in particular estimating the constants in [31], which plays a crucial role in the proof of Theorem 1.1. We believe that neither the proofs in this paper nor the ones in [31] are close to giving sharp estimates.

Our method of proof implies that for an acyclic manifold  $X$  the constant  $C$  in Theorem 1.1 can be bounded in terms of the dimension of  $X$ , and the same happens for homology

spheres; if  $X$  is a compact manifold with nonzero Euler characteristic then  $C$  can be bounded in terms of the dimension of  $X$  and the Betti numbers.

Similar comments apply to Theorem 1.3, although in this case it is easier to give concrete (albeit probably far from sharp) bounds. Instead of giving a general bound, we state two theorems giving bounds on actions on disks and even dimensional spheres. The proofs use the same ideas as the proof of Theorem 1.3, but restricting to actions on disks and spheres allows us to get stronger bounds than in the general case.

Define a map  $f : \mathbb{Z} \rightarrow \mathbb{N}$  as follows. For any nonnegative integer  $k$  let

$$f(k) = 2^k \prod_{p \geq 3} p^{[k/p]},$$

where the product is over the set of odd primes, and set  $f(k) = 1$  for every negative integer  $k$ . Note that if  $k$  is nonnegative then  $f(k)$  divides  $2^{k-[k/2]}k!$ .

**Theorem 1.5.** *Let  $n$  be a natural number and let  $X$  be the  $n$ -dimensional disk. Let  $k = [(n-3)/2]$ .*

- (1) *Any finite abelian group  $A$  acting smoothly on  $X$  has a subgroup  $A' \subseteq A$  such that  $[A : A']$  divides  $f(k)$ , and  $\chi(X^{A'}) = 1$ .*
- (2) *For any finite abelian group  $A$  acting smoothly on  $X$  such that all prime divisors  $p$  of  $|A|$  satisfy  $p > \max\{2, k\}$  we have  $\chi(X^A) = 1$ .*

**Theorem 1.6.** *Let  $m$  be a natural number and let  $X$  be a smooth  $2m$ -dimensional homology sphere.*

- (1) *Any finite abelian group  $A$  acting smoothly on  $X$  has a subgroup  $A' \subseteq A$  such that  $[A : A']$  divides  $2^{m+1}f(m-1)$ , and  $|X^{A'}| \geq 2$ .*
- (2) *For any finite abelian group  $A$  acting smoothly on  $X$  such that all prime divisors  $p$  of  $|A|$  satisfy  $p > \max\{2, m-1\}$  we have  $|X^A| \geq 2$ .*

**1.4. Notation and conventions.** We denote inclusion of sets with the symbol  $\subseteq$ ; the symbol  $\subset$  is reserved for *strict* inclusion. As usual in the theory of finite transformation groups in this paper  $\mathbb{Z}_n$  denotes  $\mathbb{Z}/n\mathbb{Z}$ , not to be mistaken, when  $n$  is a prime  $p$ , with the  $p$ -adic integers. If  $p$  is a prime we denote by  $\mathbb{F}_p$  the field of  $p$  elements. When we say that a group  $G$  can be generated by  $d$  elements we mean that there are elements  $g_1, \dots, g_d \in G$ , *not necessarily distinct*, which generate  $G$ . All manifolds appearing in the text may, unless we say the contrary, be open and have boundary. If a group  $G$  acts on a set  $X$  we denote the stabiliser of  $x \in X$  by  $G_x$ , and for any subset  $S \subset G$  we denote  $X^S = \{x \in X \mid S \subseteq G_x\}$ .

**1.5. Remark.** This paper is the result of combining substantial revisions of [26] and [28]. We have left out an entire section of [26] which gives a different proof of Theorem 1.1 for compact manifolds without odd cohomology using Theorem 1.3. This proof leads to stronger results than the geometric arguments in the present paper, but apart from requiring more restrictive hypothesis it is somewhat involved and it still uses the

classification of finite simple groups. (However, in dimension 4 the arguments of [26] allow to prove Ghys's conjecture for manifolds with nonzero Euler characteristic without using the classification of finite simple groups, see [27].)

**1.6. Contents of the paper.** Section 2 contains some preliminary results. In Section 3 we recall the main result in [31] and prove a slight strengthening of it. In Section 4 we consider the situation of an action of a finite group on a manifold preserving a submanifold on which the induced action is abelian, and prove a lemma which plays a crucial role in the proof of Theorem 1.1. In Section 5 we treat the cases of open acyclic manifolds (Theorem 5.1) and compact manifolds with nonzero Euler characteristic (Theorem 5.2), and in Section 6 we treat the case of homology spheres (Theorem 6.1). In Section 7 we introduce the notion of  $\lambda$ -stable action and its basic properties. These are used in Section 8 to prove Theorem 1.3. Finally, in Section 9 we prove Theorems 1.5 and 1.6.

**1.7. Acknowledgement.** I am very pleased to acknowledge my indebtedness to Alexandre Turull. It's thanks to him that Theorem 1.1, which in earlier versions of this paper referred only to finite solvable groups, has become a theorem on arbitrary finite groups.

## 2. PRELIMINARIES

**2.1. Local linearization of smooth finite group actions.** The following result is well known. We recall it because of its crucial role in some of the arguments of this paper. Statement (1) implies that the fixed point set of a finite group action on a manifold with boundary is a neat submanifold in the sense of §1.4 in [18].

**Lemma 2.1.** *Let a finite group  $\Gamma$  act smoothly on a manifold  $X$ , and let  $x \in X^\Gamma$ . The tangent space  $T_x X$  carries a linear action of  $\Gamma$ , defined as the derivative at  $x$  of the action on  $X$ , satisfying the following properties.*

- (1) *There exist neighborhoods  $U \subset T_x X$  and  $V \subset X$ , of  $0 \in T_x X$  and  $x \in X$  respectively, such that:*
  - (a) *if  $x \notin \partial X$  then there is a  $\Gamma$ -equivariant diffeomorphism  $\phi: U \rightarrow V$ ;*
  - (b) *if  $x \in \partial X$  then there is  $\Gamma$ -equivariant diffeomorphism  $\phi: U \cap \{\xi \geq 0\} \rightarrow V$ , where  $\xi$  is a nonzero  $\Gamma$ -invariant element of  $(T_x X)^*$  such that  $\text{Ker } \xi = T_x \partial X$ .*
- (2) *If the action of  $\Gamma$  is effective and  $X$  is connected then the action of  $\Gamma$  on  $T_x X$  is effective, so it induces an inclusion  $\Gamma \hookrightarrow \text{GL}(T_x X)$ .*
- (3) *If  $\Gamma' \triangleleft \Gamma$  and  $\dim_x X^\Gamma < \dim_x X^{\Gamma'}$  then there exists an irreducible  $\Gamma$ -submodule  $V \subset T_x X$  on which the action of  $\Gamma$  is nontrivial but the action of  $\Gamma'$  is trivial.*

*Proof.* We first construct a  $\Gamma$ -invariant Riemannian metric  $g$  on  $X$  with respect to which  $\partial X \subset X$  is totally geodesic. Take any tangent vector field on a neighborhood of  $\partial X$  whose restriction to  $\partial X$  points inward; averaging over the action of  $\Gamma$ , we get a  $\Gamma$ -invariant vector field which still points inward, and its flow at short time defines an embedding  $\psi: \partial X \times [0, \epsilon) \rightarrow X$  for some small  $\epsilon > 0$  such that  $\psi(x, 0) = x$  and  $\psi(\gamma \cdot x, t) = \gamma \cdot \psi(x, t)$

for any  $x \in \partial X$  and  $t \in [0, \epsilon)$ . Let  $h$  be a  $\Gamma$ -invariant Riemannian metric on  $\partial X$  and consider any Riemannian metric on  $X$  whose restriction to  $\psi(\partial X \times [0, \epsilon/2])$  is equal to  $h + dt^2$ . Averaging this metric over the action of  $\Gamma$  we obtain a metric  $g$  with the desired property. The exponential map with respect to  $g$  gives the local diffeomorphism in (1). To prove (2), assume that the action of  $\Gamma$  on  $X$  is effective. (1) implies that for any subgroup  $\Gamma' \subseteq \Gamma$  the fixed point set  $X^{\Gamma'}$  is a submanifold of  $X$  and that  $\dim_x X = \dim(T_x X)^{\Gamma'}$  for any  $x \in X^{\Gamma'}$ ; furthermore,  $X^{\Gamma'}$  is closed by the continuity of the action. So if some element  $\gamma \in \Gamma$  acts trivially on  $T_x X$ , then  $X^\gamma$  is a closed submanifold of  $X$  satisfying  $\dim_x X^\gamma = \dim X$ . Since  $X$  is connected this implies  $X^\gamma = X$ , so  $\gamma = 1$ , because the action of  $\Gamma$  on  $X$  is effective. Finally, (3) follows from (1) ( $V$  can be defined as any of the irreducible factors in the  $\Gamma$ -module given by the perpendicular of  $T_x X^\Gamma$  in  $T_x X^{\Gamma'}$ ).  $\square$

## 2.2. Points with big stabilizer for actions of $p$ -groups on compact manifolds.

Let  $G$  be a group and let  $\mathcal{C}$  be a simplicial complex endowed with an action of  $G$ . We say that this action is good if for any  $g \in G$  and any  $\sigma \in \mathcal{C}$  such that  $g(\sigma) = \sigma$  we have  $g(\sigma') = \sigma'$  for any subsimplex  $\sigma' \subseteq \sigma$  (equivalently, the restriction of the action of  $g$  to  $|\sigma| \subset |\mathcal{C}|$  is the identity). This property is called condition (A) in [3, Chap. III, §1]. If  $\mathcal{C}$  is a simplicial complex and  $G$  acts on  $\mathcal{C}$ , then the induced action of  $G$  on the barycentric subdivision  $\text{sd } \mathcal{C}$  is good (see [3, Chap. III, Proposition 1.1]).

Suppose that  $G$  acts on a compact manifold  $Y$ , possibly with boundary. A  $G$ -good triangulation of  $Y$  is a pair  $(\mathcal{C}, \phi)$ , where  $\mathcal{C}$  is a finite simplicial complex endowed with a good action of  $G$  and  $\phi: Y \rightarrow |\mathcal{C}|$  is a  $G$ -equivariant homeomorphism. For any smooth action of a finite group  $G$  on a manifold  $X$  there exist  $G$ -good triangulations of  $X$  (by the previous comments it suffices to prove the existence of a  $G$ -equivariant triangulation; this can be easily obtained adapting the construction of triangulations of smooth manifolds given in [6] to the finitely equivariant setting; for much more detailed results, see [19]).

**Lemma 2.2.** *Let  $Y$  be a compact smooth manifold, possibly with boundary, satisfying  $\chi(Y) \neq 0$ . Let  $p$  be a prime, and let  $G$  be a finite  $p$ -group acting smoothly on  $Y$ . Let  $r$  be the biggest nonnegative integer such that  $p^r$  divides  $\chi(X)$ . There exists some  $y \in Y$  whose stabilizer  $G_y$  satisfies  $[G : G_y] \leq p^r$ .*

*Proof.* Let  $(\mathcal{C}, \phi)$  be a  $\Gamma$ -good triangulation of  $Y$ . The cardinal of each of the orbits of  $G$  acting on  $\mathcal{C}$  is a power of  $p$ . If the cardinal of all orbits were divisible by  $p^{r+1}$ , then for each  $d$  the cardinal of the set of  $d$ -dimensional simplices in  $\mathcal{C}$  would be divisible by  $p^{r+1}$ , and consequently  $\chi(Y) = \chi(\mathcal{C})$  would also be divisible by  $p^{r+1}$ , contradicting the definition of  $r$ . Hence, there must be at least one simplex  $\sigma \in \mathcal{C}$  whose orbit has at most  $p^r$  elements. This means that the stabilizer  $G_\sigma$  of  $\sigma$  has index at most  $p^r$ . If  $y \in Y$  is a point such that  $\phi(y) \in |\sigma| \subseteq |\mathcal{C}|$ , then  $y$  is fixed by  $G_\sigma$ , because the triangulation is good.  $\square$

### 2.3. Fixed point loci of actions of abelian $p$ -groups.

**Lemma 2.3.** *Let  $X$  be a manifold, let  $p$  be a prime, and let  $G$  be a finite  $p$ -group acting continuously on  $X$ . We have*

$$\sum_j b_j(X^G; \mathbb{F}_p) \leq \sum_j b_j(X; \mathbb{F}_p).$$

*Proof.* If  $|G| = p$  then the statement follows from [2, Theorem III.4.3]. For general  $G$  use ascending induction on  $|G|$ . In the induction step, choose a central subgroup  $G_0 \subset G$  of order  $p$  and apply the inductive hypothesis to the action of  $G/G_0$  on  $X^{G_0}$ .  $\square$

### 3. TESTING JORDAN'S PROPERTY ON $\{p, q\}$ -GROUPS

Suppose that  $\mathcal{C}$  is a set of finite groups. We denote by  $\mathcal{T}(\mathcal{C})$  the set of all  $T \in \mathcal{C}$  such that there exist primes  $p$  and  $q$ , a Sylow  $p$ -subgroup  $P$  of  $T$  (which might be trivial), and a normal Sylow  $q$ -subgroup  $Q$  of  $T$ , such that  $T = PQ$ . (In particular, if  $T \in \mathcal{T}(\mathcal{C})$  then  $|T| = p^\alpha q^\beta$  for some primes  $p$  and  $q$  and nonnegative integers  $\alpha, \beta$ .)

Let  $C$  and  $d$  be positive integers. We say that a set of groups  $\mathcal{C}$  satisfies (the Jordan property)  $\mathcal{J}(C, d)$  if each  $G \in \mathcal{C}$  has an abelian subgroup  $A$  such that  $[G : A] \leq C$  and  $A$  can be generated by  $d$  elements. For convenience, we will say that  $\mathcal{C}$  satisfies the Jordan property, without specifying any constants, whenever there exist some  $C$  and  $d$  such that  $\mathcal{C}$  satisfies  $\mathcal{J}(C, d)$ .

The following is the main result in [31]:

**Theorem 3.1.** *Let  $d$  and  $M$  be positive integers. Let  $\mathcal{C}$  be a set of finite groups which is closed under taking subgroups and such that  $\mathcal{T}(\mathcal{C})$  satisfies  $\mathcal{J}(M, d)$ . Then there exists a positive integer  $C_0$  such that  $\mathcal{C}$  satisfies  $\mathcal{J}(C_0, d)$ .*

We next prove a refinement of this theorem. Given a set of finite groups  $\mathcal{C}$ , we define  $\mathcal{T}_A(\mathcal{C})$  exactly like  $\mathcal{T}(\mathcal{C})$  but imposing additionally that the Sylow subgroups are abelian (in particular,  $\mathcal{T}_A(\mathcal{C}) \subset \mathcal{T}(\mathcal{C})$ ). In concrete terms, a group  $G \in \mathcal{C}$  belongs to  $\mathcal{T}_A(\mathcal{C})$  if and only if there exist primes  $p$  and  $q$ , an abelian Sylow  $p$ -subgroup  $P \subseteq G$  and a normal abelian Sylow  $q$ -subgroup  $Q \subseteq G$ , such that  $G = PQ$ . Denote by  $\mathcal{P}(\mathcal{C})$  the set of groups  $G \in \mathcal{C}$  with the property that there exists a prime  $p$  such that  $G$  is a  $p$ -group.

If  $G$  is a (possibly infinite) group we denote by  $\mathcal{C}(G)$  the set of finite subgroups of  $G$  and we let  $\mathcal{T}_A(G) := \mathcal{T}_A(\mathcal{C}(G))$ .

**Corollary 3.2.** *Let  $d$  and  $M$  be positive integers. Let  $\mathcal{C}$  be a set of finite groups which is closed under taking subgroups and such that  $\mathcal{P}(\mathcal{C}) \cup \mathcal{T}_A(\mathcal{C})$  satisfies  $\mathcal{J}(M, d)$ . Then there exists a positive integer  $C_0$  such that  $\mathcal{C}$  satisfies  $\mathcal{J}(C_0, d)$ .*

*Proof.* Let  $d$  and  $M$  be positive integers, suppose that  $\mathcal{C}$  is a set of finite groups which closed under taking subgroups, and assume that  $\mathcal{P}(\mathcal{C}) \cup \mathcal{T}_A(\mathcal{C})$  satisfies  $\mathcal{J}(M, d)$ . Let  $C := M^2(M!)^d$ . We claim that  $\mathcal{T}(\mathcal{C})$  satisfies  $\mathcal{J}(C, d)$ . This immediately implies our



result, in view of Theorem 3.1. Since  $\mathcal{P}(\mathcal{C})$  satisfies  $\mathcal{J}(M, d)$ , to justify the claim it suffices to prove the following fact.

Let  $G$  be a finite group, let  $p, q$  be distinct prime numbers, let  $P \subseteq G$  be a  $p$ -Sylow subgroup, let  $Q \subseteq G$  be a normal  $q$ -Sylow subgroup, and assume that  $G = PQ$ ; if there exist abelian subgroups  $P_0 \subseteq P$  and  $Q_0 \subseteq Q$  such that  $[P : P_0] \leq M$ ,  $[Q : Q_0] \leq M$ , and  $Q_0$  can be generated by  $d$  elements, then there exists some  $G' \in \mathcal{T}_A(G)$  such that  $[G : G'] \leq C$ .

To prove this, define  $Q' := \bigcap_{\phi \in \text{Aut}(Q)} \phi(Q_0)$ , where  $\text{Aut}(Q)$  denotes the group of automorphisms of  $Q$ . Clearly  $Q'$  is an abelian characteristic subgroup of  $Q$ , so it is normal in  $G$  (because  $Q$  is normal in  $G$ ). Define  $G' := P_0 Q'$ . Then  $G' \in \mathcal{T}_A(G)$ , so we only need to prove that  $[G : G'] \leq C$ .

Suppose that  $\{g_1, \dots, g_\delta\}$  is a generating set of  $Q_0$  such that  $\delta \leq d$  and  $Q_0 \simeq \prod_j \langle g_j \rangle$ , where  $\langle g_j \rangle \subseteq Q_0$  is the subgroup generated by  $g_j$  (such generating set exists because  $Q_0$  is abelian and can be generated by  $d$  elements). If  $\Gamma \subseteq Q_0$  is any subgroup of index at most  $M$ , then  $g_j^{M!}$  belongs to  $\Gamma$  for each  $j$ . Consequently, the subgroup  $Q'' \subseteq Q_0$  generated by  $\{g_1^{M!}, \dots, g_\delta^{M!}\}$  is contained in any subgroup  $\Gamma \subseteq Q_0$  of index at most  $M$ . In particular  $Q'' \subseteq Q'$ , because for any  $\phi \in \text{Aut}(Q)$  we have  $[Q_0 : Q_0 \cap \phi(Q_0)] \leq M$ . On the other hand,  $[Q_0 : Q''] \leq (M!)^\delta \leq (M!)^d$ , so a fortiori  $[Q_0 : Q'] \leq (M!)^d$ . Since  $[G : G'] = [P : P_0][Q : Q'] = [P : P_0][Q : Q_0][Q_0 : Q']$ , the result follows.  $\square$

#### 4. ACTIONS OF FINITE GROUPS ON REAL VECTOR BUNDLES

Define, for any smooth manifold  $Y$ ,

$$\mathcal{T}_A(Y) := \mathcal{T}_A(\text{Diff}(Y)).$$

Suppose that  $E \rightarrow Y$  be a smooth real vector bundle. Denote by  $\text{Diff}(E \rightarrow Y)$  the group of smooth bundle automorphisms lifting arbitrary diffeomorphisms of the base  $Y$  (equivalently, diffeomorphisms of  $E$  which map fibers to fibers and whose restriction to each fiber is a linear map). Denote by

$$(2) \quad \pi : \text{Diff}(E \rightarrow Y) \rightarrow \text{Diff}(Y)$$

the map which assigns to each  $\phi \in \text{Diff}(E \rightarrow Y)$  the diffeomorphism of  $\psi \in \text{Diff}(Y)$  such that  $\phi(E_y) = \phi(E_{\psi(y)})$  for every  $y$ , where  $E_y$  is the fiber of  $E$  over  $y$ .

Define also

$$\mathcal{T}_A(E \rightarrow Y) := \mathcal{T}_A(\text{Diff}(E \rightarrow Y)).$$

Since the properties defining the groups in  $\mathcal{T}_A(\mathcal{C})$  in Section 3 are preserved by passing to quotients, for any  $G \in \mathcal{T}_A(E \rightarrow Y)$  we have  $\pi(G) \in \mathcal{T}_A(Y)$ .

**Lemma 4.1.** *Assume that  $Y$  is connected and let  $r$  be the rank of  $E$ . Suppose that  $G \in \mathcal{T}_A(E \rightarrow Y)$  and that  $\pi(G) \in \mathcal{T}_A(Y)$  is abelian. Then  $G$  has an abelian subgroup  $A \subseteq G$  satisfying  $[G : A] \leq r!$ .*

*Proof.* Take a group  $G \in \mathcal{T}_A(E \rightarrow Y)$  such that  $\pi(G)$  is abelian. There exist two distinct primes  $p$  and  $q$ , a  $p$ -Sylow subgroup  $P \subseteq G$ , and a normal  $q$ -Sylow subgroup  $Q \subset G$ , such that  $G = PQ$ . Furthermore, both  $P$  and  $Q$  are abelian. Let  $Q_0 = Q \cap \text{Ker } \pi \subseteq Q$ . Then  $Q_0$  is normal in  $G$ , since  $Q_0 = Q \cap (G \cap \text{Ker } \pi)$  and both  $Q$  and  $G \cap \text{Ker } \pi$  are normal in  $G$ . So the action of  $P$  on  $G$  given by conjugation preserves both  $Q$  and  $Q_0$ . Furthermore, since  $\pi(G)$  is abelian we have, for any  $\gamma \in P$  and  $\eta \in Q$ ,  $\pi(\gamma\eta\gamma^{-1}\eta^{-1}) = \pi(\gamma)\pi(\eta)\pi(\gamma)^{-1}\pi(\eta)^{-1} = 1$ , which is equivalent to  $\gamma\eta\gamma^{-1}\eta^{-1} \in Q_0$ .

The complexified vector bundle  $E \otimes \mathbb{C}$  splits as a direct sum of subbundles indexed by the characters of  $Q_0$ ,

$$(3) \quad E \otimes \mathbb{C} = \bigoplus_{\rho \in \text{Mor}(Q_0, \mathbb{C}^*)} E_\rho,$$

where  $v \in E_\rho$  if and only if  $\eta \cdot v = \rho(\eta)v$  for any  $\eta \in Q_0$ . The action of  $P$  on  $E \otimes \mathbb{C}$  permutes the summands  $\{E_\rho\}$ . In concrete terms, if we define for any  $\rho \in \text{Mor}(Q_0, \mathbb{C}^*)$  and  $\gamma \in P$  the character  $\rho_\gamma \in \text{Mor}(Q_0, \mathbb{C}^*)$  by  $\rho_\gamma(\eta) = \rho(\gamma^{-1}\eta\gamma)$  for any  $\eta \in Q_0$ , then we have  $\gamma \cdot E_\rho = E_{\rho_\gamma}$ . Since there are at most  $r$  nonzero summands in (3) (because  $Y$  is connected and the rank of  $E$  is  $r$ ), the subgroup  $P' \subseteq P$  consisting of those elements which preserve each nonzero subbundle  $E_\rho$  satisfies  $[P : P'] \leq r!$ . Furthermore, each element of  $P'$  commutes with all the elements in  $Q_0$ , because  $P'$  acts linearly on  $E \otimes \mathbb{C}$  preserving the summands in (3) and the action of  $Q_0$  on each summand is given by homothecies (in particular, the action of each element of  $Q_0$  lifts the identity on  $Y$ ). Hence, the action of  $P'$  on  $Q$  given by conjugation gives a morphism

$$P' \rightarrow B := \{\phi \in \text{Aut}(Q) \mid \phi(\eta) = \eta \text{ for each } \eta \in Q_0, \quad \phi(\eta)\eta^{-1} \in Q_0 \text{ for each } \eta \in Q\}.$$

We now prove that  $B$  is  $q$ -group. Let  $\phi \in B$  be any element, and define a map  $f : Q \rightarrow Q_0$  by the condition that  $f(\eta) = \phi(\eta)\eta^{-1}$  for every  $\eta$ . Since  $Q$  is abelian and  $\phi$  is a morphism of groups,  $f$  is also a morphism of groups. Furthermore,  $f(f(\eta)) = 1$  for every  $\eta \in Q$ , because  $Q_0 \subseteq \text{Ker } f$ . Using induction it follows that  $\phi^k(\eta) = f(\eta)^k\eta$  for every  $k \in \mathbb{N}$ . Hence the order of  $\phi \in B$  divides  $|Q_0|$ , which is a power of  $q$ .

Since  $P'$  is a  $p$ -group and  $p \neq q$ , any morphism  $P' \rightarrow B$  is trivial. This implies that  $P'$  commutes with  $Q$ . Setting  $A := P'Q$ , the result follows.  $\square$

**Lemma 4.2.** *Suppose that  $X$  is a smooth connected manifold and that  $G \in \mathcal{T}_A(X)$ . Assume that there is a  $G$ -invariant connected submanifold  $Y \subseteq X$ . Let  $G_Y \subseteq \text{Diff}(Y)$  be the group consisting of all diffeomorphisms of  $Y$  which are induced by restricting to  $Y$  the action of the elements of  $G$ . Let  $r := \dim X - \dim Y$ . If  $G_Y$  is abelian, then there is an abelian subgroup  $A \subseteq G$  satisfying  $[G : A] \leq r!$ .*

*Proof.* There is an inclusion of vector bundles  $TY \hookrightarrow TX|_Y$ . Consider the quotient bundle  $E := TX|_Y / TY \rightarrow Y$ , which is the normal bundle of the inclusion  $Y \hookrightarrow X$ . Let  $\text{Diff}(X, Y)$  be the group of diffeomorphisms of  $X$  which preserve  $Y$ . There is a natural restriction map  $\rho : \text{Diff}(X, Y) \rightarrow \text{Diff}(E \rightarrow Y)$  given by restricting the diffeomorphisms in  $\text{Diff}(X, Y)$  to the first jet of the inclusion  $Y \hookrightarrow X$ , which gives a bundle automorphism  $TX|_Y \rightarrow TX|_Y$  preserving  $TY$ , and then projecting to an automorphism of  $E$ .

Furthermore, if  $\Gamma \subseteq \text{Diff}(X, Y)$  is a finite group then by (2) in Lemma 2.1 the restriction  $\rho|_{\Gamma} : \Gamma \rightarrow \rho(\Gamma)$  is injective. Applying this to the group  $G \in \mathcal{T}_A(X)$  in the statement of the lemma we obtain a group  $G_E := \rho(G) \in \mathcal{T}_A(E \rightarrow Y)$  which is isomorphic to  $G$ . Furthermore, if  $\pi : \text{Diff}(E \rightarrow Y) \rightarrow \text{Diff}(Y)$  is the map (2), then  $\pi(G_E) \in \mathcal{T}_A(Y)$  coincides with  $G_Y$ , which by hypothesis is abelian. We are thus in the setting of Lemma 4.1, so we deduce that  $G_E$  (and hence  $G$ ) has an abelian subgroup of index at most  $r!$ .  $\square$

## 5. ACTIONS OF FINITE GROUPS ON ACYCLIC MANIFOLDS AND ON COMPACT MANIFOLDS WITH $\chi \neq 0$

**Theorem 5.1.** *For any  $n$  there is a constant  $C$  such that any finite group acting smoothly and effectively on an acyclic smooth  $n$ -dimensional manifold  $X$  has an abelian subgroup of index at most  $C$ .*

*Proof.* Fix  $n$  and let  $X$  be an acyclic smooth  $n$ -dimensional manifold. Let  $\mathcal{C}$  be the set of all finite subgroups of  $\text{Diff}(X)$ . By Corollary 3.2 it suffices to prove that  $\mathcal{P}(\mathcal{C}) \cup \mathcal{T}_A(\mathcal{C})$  satisfies the Jordan property. We prove it first for  $\mathcal{P}(\mathcal{C})$  and then for  $\mathcal{T}_A(\mathcal{C})$ .

Let  $G \in \mathcal{P}(\mathcal{C})$  be a finite  $p$ -group, where  $p$  is a prime. By Smith theory the fixed point set  $X^G$  is  $\mathbb{F}_p$ -acyclic, hence nonempty (see [2, Corollary III.4.6] for the case  $G = \mathbb{Z}_p$  and use induction on  $|G|$  for the general case, as in the proof of Lemma 2.3). Let  $x \in X^G$ . By (2) in Lemma 2.1, linearizing the action of  $G$  at  $x$  we get an injective morphism  $G \hookrightarrow \text{GL}(T_x X) \simeq \text{GL}(n, \mathbb{R})$ . It follows from Jordan's Theorem 1.2 that there is an abelian subgroup  $A \subseteq G$  such that  $[G : A] \leq C_n$ , where  $C_n$  depends only on  $n$ . Furthermore, since  $A$  can be identified with a subgroup of  $\text{GL}(n, \mathbb{R})$ , it can be generated by at most  $n$  elements. We have thus proved that  $\mathcal{P}(\mathcal{C})$  satisfies  $\mathcal{J}(C_n, n)$ . The same argument also proves that any elementary  $p$ -group acting effectively on  $X$  has rank at most  $n$ .

Now let  $G \in \mathcal{T}_A(\mathcal{C})$ . By definition, there are two distinct primes  $p$  and  $q$ , an abelian  $p$ -Sylow subgroup  $P \subseteq G$ , and a normal abelian  $q$ -Sylow subgroup  $Q \subseteq G$ , such that  $G = PQ$ . Let  $Y := X^Q$ . By Smith theory,  $Y$  is a  $\mathbb{F}_q$ -acyclic manifold (combine again [2, Corollary III.4.6] with induction on  $|Q|$  as before); in particular,  $Y$  is nonempty and connected. Since  $Q$  is normal in  $G$ , the action of  $G$  on  $X$  preserves  $Y$ . Finally, since the elements of  $Q$  act trivially on  $Y$ , the action of  $G$  on  $Y$  given by restriction defines an abelian subgroup of  $\text{Diff}(Y)$ . This means that we are in the setting of Lemma 4.2, and we can deduce that  $G$  has an abelian subgroup  $A \subseteq G$  of index at most  $n!$ . Since, as explained in the previous paragraph, any elementary  $p$ -group acting on  $X$  has rank at most  $n$ ,  $A$  can be generated by at most  $n$  elements. We have thus proved that  $\mathcal{T}_A(\mathcal{C})$  satisfies  $\mathcal{J}(n!, n)$ , and the proof of the theorem is now complete.  $\square$

**Theorem 5.2.** *Let  $X$  be a compact connected smooth manifold, possibly with boundary, and satisfying  $\chi(X) \neq 0$ . There exists a constant  $C$  such that any finite group acting smoothly and effectively on  $X$  has an abelian subgroup of index at most  $C$ .*

*Proof.* Let  $\mathcal{C}$  be the set of all finite subgroups of  $\text{Diff}(X)$ . We will again deduce the theorem from Corollary 3.2, so we only need to prove that  $\mathcal{P}(\mathcal{C}) \cup \mathcal{T}_A(\mathcal{C})$  satisfies the Jordan property.

To prove that  $\mathcal{P}(\mathcal{C})$  satisfies the Jordan property, let  $s := q^e$  be the biggest prime power dividing  $\chi(X)$ . Let  $p$  be any prime, and consider a  $p$ -group  $G \in \mathcal{P}(G)$ . By Lemma 2.2, there is a point  $x \in X$  whose stabiliser  $G_x$  satisfies  $[G : G_x] \leq p^r$ , where  $p^r$  divides  $\chi(X)$ . In particular,  $[G : G_x] \leq s$ . By Lemma 2.1 there is an inclusion  $G_x \hookrightarrow \text{GL}(T_x X) \simeq \text{GL}(n, \mathbb{R})$ , where  $n = \dim X$ . By the same argument as in the proof of Theorem 5.1, there is an abelian subgroup  $A \subseteq G_x$  of index  $[G_x : A] \leq C_n$  (with  $C_n$  depending only on  $n$ ) and which can be generated by  $n$  elements. Hence,  $\mathcal{P}(\mathcal{C})$  satisfies  $\mathcal{J}(sC_n, n)$ .

We now prove that  $\mathcal{T}_A(\mathcal{C})$  satisfies the Jordan property. Let  $G \in \mathcal{T}_A(\mathcal{C})$ . There exist two distinct primes  $p$  and  $q$ , an abelian  $p$ -Sylow subgroup  $P \subseteq G$ , and a normal abelian  $q$ -Sylow subgroup  $Q \subseteq G$ , such that  $G = PQ$ . By the arguments used before (involving Lemma 2.2) there is a point  $x \in X$  whose stabiliser  $Q_x$  satisfies  $[Q : Q_x] \leq s$  for some positive integer  $s$  depending only on  $\chi(X)$ . Since  $Q_x$  is abelian and we have an inclusion  $Q_x \hookrightarrow \text{GL}(T_x X)$ , we know that  $Q_x$  can be generated by  $n$  elements. Define

$$Q' := \bigcap_{\phi \in \text{Aut}(Q)} \phi(Q_x).$$

By the arguments at the end of the proof of Corollary 3.2, we have  $[Q_x : Q'] \leq (s!)^n$ . Since  $Q' \subset Q_x$ , we have  $x \in X^{Q'}$ , so  $X^{Q'}$  is nonempty. Also,  $Q'$  does not contain any elementary  $q$ -group of rank greater than  $n$  (because it is a subgroup of  $G_x$ ), so it can be generated by  $n$  elements. By Lemma 2.3 we have

$$\sum_j b_j(X^{Q'}; \mathbb{F}_q) \leq \sum_j b_j(X; \mathbb{F}_q) \leq K := \sum_j \max\{b_j(X; \mathbb{F}_p) \mid p \text{ prime}\},$$

where  $K$  is finite because  $X$  is compact. In particular,  $X^{Q'}$  has at most  $K$  connected components. On the other hand,  $Q'$  is a characteristic subgroup of  $Q$ , and since  $Q$  is normal in  $G$  it follows that  $Q'$  is also normal in  $G$ . Hence the action of  $P$  on  $X$  preserves  $X^{Q'}$ . Since the latter has at most  $K$  connected components, there exists a subgroup  $P_0 \subseteq P$  of index  $[P : P_0] \leq K$  and a connected component  $Y \subseteq X^{Q'}$  such that  $P_0$  preserves  $Y$ . By Lemma 2.2 there is also a subgroup  $P' \subseteq P_0$  which fixes some point in  $X$  and such that  $[P_0 : P'] \leq s$ , which implies as before that  $P'$  can be generated by at most  $n$  elements. Let  $G' := P'Q'$ . We can bound

$$[G : G'] = [P : P_0][P_0 : P'][Q : Q_x][Q_x : Q'] \leq sKs(s!)^n.$$

On the other hand,  $G'$  preserves  $Y$ , and its induced action on  $Y$  is abelian (because  $Q'$  acts trivially on  $Y$  and  $P'$  is abelian). By Lemma 4.2, there exists an abelian subgroup  $G'' \subseteq G'$  satisfying  $[G' : G''] \leq n!$ . It follows that

$$[G : G''] = [G : G'][G' : G''] \leq M := sKs(s!)^n n!.$$

Since both  $P'$  and  $Q'$  can be generated by at most  $n$  elements,  $G'$  does not contain any elementary  $p$ -group or  $q$ -group of rank greater than  $n$ , which implies that  $G''$  can be generated by  $n$  elements. We have thus proved that  $\mathcal{T}_A(\mathbb{C})$  satisfies  $\mathcal{J}(M, n)$ , so the proof of the theorem is complete.  $\square$

## 6. ACTIONS ON HOMOLOGY SPHERES

Recall some standard terminology: given a ring  $R$  and an integer  $n \geq 0$ , an  $R$ -homology  $n$ -sphere is a topological  $n$ -manifold<sup>3</sup>  $M$  satisfying  $H_*(M; R) \simeq H_*(S^n; R)$ . A homology  $n$ -sphere is a  $\mathbb{Z}$ -homology  $n$ -sphere. By the universal coefficient theorem any homology  $n$ -sphere is a  $\mathbb{Z}_p$ -homology  $n$ -sphere for any prime  $p$ . Standard properties of topological manifolds imply that for any prime  $p$  and any integer  $n$  any  $\mathbb{Z}_p$ -homology  $n$ -sphere is compact and orientable.

**Theorem 6.1.** *For any  $n$  there is a constant  $C$  such that any finite group acting smoothly and effectively on a smooth homology  $n$ -sphere has an abelian subgroup of index at most  $C$ .*

Before proving the theorem we collect a few facts which will be used in our arguments. In what follows  $p$  denotes an arbitrary prime.

Let  $p$  be any prime, and let  $G$  be a finite  $p$ -group acting on a  $\mathbb{Z}_p$ -homology  $n$ -sphere  $S$ . Then the fixed point set  $S^G$  is a  $\mathbb{Z}_p$ -homology  $n(G)$ -sphere for some integer  $-1 \leq n(G) \leq n$ , where a  $(-1)$ -sphere is by convention the empty set (see [2, IV.4.3] for the case  $G = \mathbb{Z}_p$ ; the general case follows by induction, see [2, IV.4.5]). Furthermore, if  $S$  is smooth and the action of  $G$  is smooth and nontrivial, then  $S^G$  is a smooth proper submanifold of  $S$  and  $n(G) < n$ .

Suppose that  $G \simeq (\mathbb{Z}_p)^r$  is an elementary abelian  $p$ -group acting on a  $\mathbb{Z}_p$ -homology  $n$ -sphere  $S$ . The following formula ([2, Theorem XIII.2.3]) was proved by Borel:

$$(4) \quad n - n(G) = \sum_{\substack{H \subseteq G \text{ subgroup} \\ [G:H]=p}} (n(H) - n(G)).$$

Now suppose that  $G$  is a finite  $p$ -group acting on a smooth  $\mathbb{Z}_p$ -homology  $n$ -sphere  $S$ . Dotzel and Hamrick proved in [10] that there exists a real representation of  $\rho : G \rightarrow \mathrm{GL}(V)$  such that  $\dim_{\mathbb{R}} V^H = n(H) + 1$  for each subgroup  $H \subseteq G$ . This implies that  $\dim_{\mathbb{R}} V = n + 1$  (take  $H = \{1\}$ ) and that, if the action is effective,  $\rho$  is injective (because for any nontrivial  $H \subseteq G$  we have  $n(H) < n$ ). Consequently, if  $G$  acts effectively on  $S$  then we can identify  $G$  with a subgroup of  $\mathrm{GL}(n + 1, \mathbb{R})$  (which, when  $S = S^n$ , does not mean that the original action of  $G$  on  $S^n$  is necessarily linear!). In particular, if  $G \simeq (\mathbb{Z}_p)^r$  acts effectively on  $S$ , then  $r \leq n + 1$ . Alternatively,  $r$  can be bounded using a general theorem of Mann and Su [22].

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<sup>3</sup>This definition of homology sphere is more restrictive than that in [2] or [21], where homology spheres are not required to be topological manifolds.

Let  $p$  be now an odd prime, let  $d$  be a positive integer, and consider a morphism  $\psi : \mathbb{Z}_d \rightarrow \text{Aut}(\mathbb{Z}_p)$ . Suppose that the image under  $\psi$  of a generator  $g \in \mathbb{Z}_d$  is multiplication by some  $y \in \mathbb{Z}_p^*$  (here we use additive notation on  $\mathbb{Z}_p$ ). The following is a very slight modification of a result of Guazzi and Zimmermann [16, Lemma 2]:

**Lemma 6.2.** *If  $\mathbb{Z}_p \rtimes_{\psi} \mathbb{Z}_d$  acts effectively on a smooth  $\mathbb{Z}_p$ -homology  $n$ -sphere  $S$  and the restriction of the action to  $\mathbb{Z}_p$  is free then  $y^{n+1} = 1 \in \mathbb{Z}_p^*$*

The original result of Guazzi and Zimmermann does not require the restriction of the action to  $\mathbb{Z}_p$  to be free, but it requires the action of  $\mathbb{Z}_p \rtimes_{\psi} \mathbb{Z}_d$  to be orientation preserving. The proof we give of Lemma 6.2 is essentially the same as [16, Lemma 2]; we give it to justify that the result is valid without assuming that  $\mathbb{Z}_p \rtimes_{\psi} \mathbb{Z}_d$  acts orientation-preservingly.

*Proof.* Since  $S$  is smooth and compact,  $p$  is odd, and the action of  $\mathbb{Z}_p$  is free,  $n$  must be odd, say  $n = 2\nu + 1$ . Let  $\pi : S_{\mathbb{Z}_p} \rightarrow B\mathbb{Z}_p$  be the Borel construction and let  $\zeta : S/\mathbb{Z}_p \rightarrow B\mathbb{Z}_p$  be the composition of a homotopy equivalence  $S/\mathbb{Z}_p \simeq S_{\mathbb{Z}_p}$  (which exists because  $\mathbb{Z}_p$  acts freely on  $S$ ) with  $\pi$ . A simple argument using the Serre spectral sequence for  $\zeta$  proves that the map  $H^k(\zeta) : H^k(B\mathbb{Z}_p; \mathbb{Z}_p) \rightarrow H^k(S/\mathbb{Z}_p; \mathbb{Z}_p)$  is an isomorphism for  $0 \leq k \leq n$  (note that, since  $p$  is odd, the action of  $\mathbb{Z}_p$  on  $S$  is orientation preserving, so the second page of the Serre spectral sequence for  $\zeta$  has entries  $H^u(B\mathbb{Z}_p; \mathbb{Z}_p) \otimes H^v(S; \mathbb{Z}_p)$ ). We have  $H^*(B\mathbb{Z}_p; \mathbb{Z}_p) \simeq \Lambda(\alpha) \otimes \mathbb{Z}_p[\beta]$  where  $\deg \alpha = 1$ ,  $\deg \beta = 2$ , and  $\beta = b(\alpha)$ , where  $b$  denotes the Bockstein. The action of  $\mathbb{Z}_d$  on  $\mathbb{Z}_p$  induces an action  $\phi^* : \mathbb{Z}_d \rightarrow \text{Aut}(H^*(B\mathbb{Z}_p; \mathbb{Z}_p))$  satisfying  $\phi^*(g)(\alpha) = y\alpha$ , and by naturality and linearity of the Bockstein we also have  $\phi^*(g)(\beta) = y\beta$ . This implies that the action of  $g$  on  $H^n(B\mathbb{Z}_p; \mathbb{Z}_p) = \mathbb{Z}_p \langle \alpha\beta^{\nu} \rangle$  is multiplication by  $y^{1+\nu}$ . Since the map  $\zeta$  is  $\mathbb{Z}_d$ -equivariant, the action of  $g$  on  $H^n(S/\mathbb{Z}_p; \mathbb{Z}_p)$  is also multiplication by  $y^{1+\nu}$ . This is the reduction mod  $p$  of the action of  $g$  on  $H^n(S/\mathbb{Z}_p; \mathbb{Z}) \simeq \mathbb{Z}$  (the isomorphism follows from the fact that  $S$  is compact and orientable, and that the action of  $\mathbb{Z}_p$  is orientation preserving). Since the action of  $g$  is by a diffeomorphism, it follows that  $y^{1+\nu}$  is the reduction mod  $p$  of  $\pm 1$ , so  $y^{2(1+\nu)} = y^{n+1} = 1 \in \mathbb{Z}_p^*$ .  $\square$

**6.1. Proof of Theorem 6.1.** Fix some  $n \geq 1$ , let  $S$  be a smooth homology  $n$ -sphere, and let  $\mathcal{C}$  be the set of finite subgroups of  $\text{Diff}(S)$ . We are going to use Corollary 3.2, so we need to prove that  $\mathcal{P}(\mathcal{C}) \cup \mathcal{T}_A(\mathcal{C})$  satisfies the Jordan property. As in the proofs of the other two theorems of this paper, we treat separately  $\mathcal{P}(\mathcal{C})$  and  $\mathcal{T}_A(\mathcal{C})$ .

If  $G \in \mathcal{P}(\mathcal{C})$  then by the theorem of Dotzel and Hamrick [10] we may identify  $G$  with a subgroup of  $\text{GL}(n+1, \mathbb{R})$ . By Theorem 1.2 there is an abelian subgroup  $A \subseteq G$  of index at most  $C_{n+1}$ . Furthermore,  $A$  can be generated by  $n+1$  elements. Consequently,  $\mathcal{P}(\mathcal{C})$  satisfies  $\mathcal{J}(C_{n+1}, n+1)$ .

The fact that  $\mathcal{T}_A(\mathcal{C})$  satisfies the Jordan property is a consequence of the following lemma, combined with the existence of a uniform upper bound, for any prime  $p$ , on the rank of elementary  $p$ -groups acting effectively on  $S^n$  (such bound follows, as we said, either from the theorem of Dotzel and Hamrick [10] or from that of Mann and Su [22]).

**Lemma 6.3.** *Given two integers  $m \geq 0, r \geq 1$  there exists an integer  $K_{m,r} \geq 1$  such that for any two distinct primes  $p$  and  $q$ , any abelian  $p$ -group  $P$  of rank at most  $r$ , any abelian  $q$ -group  $Q$ , any morphism  $\phi : P \rightarrow \text{Aut}(Q)$ , any smooth  $\mathbb{Z}_q$ -homology  $m$ -sphere  $S$ , and any smooth and effective action of  $G := Q \rtimes_{\phi} P$  on  $S$ , there is an abelian subgroup  $A \subseteq G$  satisfying  $[G : A] \leq K_{m,r}$ .*

*Proof.* Fix some integer  $r \geq 1$ . We prove the lemma, for this fixed value of  $r$ , using induction on  $m$ . The case  $m = 0$  being obvious, we may suppose that  $m > 0$  and assume that Lemma 6.3 is true for smaller values of  $m$ . Let  $p, q, P, Q, \phi, G, S$  be as in the statement of the lemma, and take a smooth effective action of  $G$  on  $S$ .

Suppose first that  $S^Q \neq \emptyset$ . Then  $S^Q$  is a smooth  $\mathbb{Z}_q$ -homology sphere of smaller dimension than  $S$ . Furthermore, since  $Q$  is normal in  $G$ , the action of  $G$  on  $S$  preserves  $S^Q$ . Let  $G_0 \subseteq \text{Diff}(S^Q)$  be the diffeomorphisms of  $S^Q$  induced by restricting the action of the elements of  $G$  on  $S$  to  $S^Q$ . Then  $G_0$  is a quotient of  $G$ , and this implies that  $G_0 \simeq Q_0 \rtimes P_0$ , where  $P_0$  (resp.  $Q_0$ ) is a quotient of  $P$  (resp.  $Q$ ). Hence we may apply the inductive hypothesis to the action of  $G_0$  on  $S^Q$  and deduce that there exists an abelian subgroup  $A_0 \subseteq G_0$  satisfying  $[G_0 : A_0] \leq K_{m-1,r}$ . Let  $\pi : G \rightarrow G_0$  be the quotient map (i.e., the restriction of the action to  $S^Q$ ), and let  $G' := \pi^{-1}(A_0) \subseteq G$ . Then  $[G : G'] \leq K_{m-1,r}$  and the action of  $G'$  satisfies the hypothesis of Lemma 4.2 with  $X = S$  and  $Y = S^Q$ . Hence, there is an abelian subgroup  $A \subseteq G'$  satisfying  $[G' : A] \leq m!$ . By the previous estimates  $A$  is an abelian subgroup of  $G$  of index  $[G : A] \leq m!K_{m-1,r}$ .

Now assume that  $S^Q = \emptyset$ . Let  $Q' := \{\eta \in Q \mid \eta^q = 1\} \subseteq Q$ . Then  $Q' \simeq (\mathbb{Z}_q)^l$ . Since  $Q'$  is a characteristic subgroup of  $Q$  and  $Q$  is normal in  $G$ ,  $Q'$  is normal in  $G$ . We distinguish two cases.

Suppose first that  $l \geq 2$ . The Borel formula (4) applied to the action of  $Q'$  on  $S$  gives

$$m + 1 = \sum_{\substack{H \subseteq Q' \text{ subgroup} \\ [Q' : H] = p}} (n(H) + 1).$$

All summands on the RHS are nonnegative integers. So at least one summand is strictly positive, and there are at most  $m + 1$  strictly positive summands. Hence, the set

$$\mathcal{H} := \{H \text{ subgroup of } Q' \mid S^H \neq \emptyset, [Q' : H] = q\}$$

is nonempty and has at most  $m + 1$  elements. The action of  $G$  on  $Q'$  by conjugation permutes the elements of  $\mathcal{H}$ , so there is a subgroup  $G' \subset G$  fixing some element  $H \in \mathcal{H}$  and such that  $[G : G'] \leq m + 1$ . The fact that  $G'$  fixes  $H$  as an element of  $\mathcal{H}$  means that  $H$  is a normal subgroup of  $G'$ , so the action of  $G'$  on  $S$  preserves  $S^H \neq \emptyset$ . We now proceed along similar lines to the previous case. Let  $G'' \subseteq \text{Diff}(S^H)$  be the diffeomorphisms of  $S^H$  induced by restricting the action of the elements of  $G'$  on  $S$  to  $S^H$ . Then  $G''$  is a quotient of  $G'$  and  $S^H$  is a smooth  $\mathbb{Z}_q$ -homology sphere of dimension strictly smaller than  $m$ ; we may thus apply the inductive hypothesis and deduce the existence of an abelian subgroup  $A' \subseteq G''$  of index at most  $K_{m-1,r}$ . Letting  $G_a \subseteq G'$  be the preimage of  $A'$  under the projection map  $G' \rightarrow G''$  we apply Lemma 4.2 to the action of  $G_a$  near the

submanifold  $S^H \subseteq S$  and conclude that  $G_a$  has an abelian subgroup  $A$  of index at most  $m!$ . Then  $[G : A] \leq (m+1)m!K_{m-1,r}$ .

Finally, suppose that  $l = 1$ . In this case  $Q'$  acts freely on  $S$  and  $Q \simeq (\mathbb{Z}_{q^s})$ . If  $q = 2$  then  $|\text{Aut}(\mathbb{Z}_{q^s})| = (q-1)q^{s-1}$  is equal to  $2^{s-1}$  and since  $p \neq 2$  the morphism  $\phi : P \rightarrow \text{Aut}(Q)$  is trivial, which means that  $G$  is abelian and there is nothing to prove. Assume then that  $q$  is odd. Taking an isomorphism  $P \simeq \mathbb{Z}_{p^{e_1}} \times \cdots \times \mathbb{Z}_{p^{e_c}}$  (with  $c \leq r$ , by our assumption on the  $p$ -rank of  $P$ ) we may apply Lemma 6.2 to the restriction of  $\phi$  to each summand,  $\phi|_{\mathbb{Z}_{p^{e_i}}} : \mathbb{Z}_{p^{e_i}} \rightarrow \text{Aut}(\mathbb{Z}_q)$  and conclude that there is a subgroup  $\Gamma_i \subset \mathbb{Z}_{p^{e_i}}$  of index at most  $m+1$  such that  $\phi(\Gamma_i)$  contains only the trivial automorphism of  $\mathbb{Z}_q$ , i.e.,  $\Gamma_i$  commutes with  $Q' \simeq \mathbb{Z}_q$ . Let  $P_0 := \Gamma_1 \times \cdots \times \Gamma_c$ . Then  $[P : P_0] \leq (m+1)^c$ . Finally, since, if we identify  $\mathbb{Z}_q$  with the  $q$ -torsion of  $\mathbb{Z}_{q^s}$ , the group

$$\{\alpha \in \text{Aut}(\mathbb{Z}_{q^s}) \mid \alpha(t) = t \text{ for every } t \in \mathbb{Z}_q\}$$

is a  $q$ -group, it follows that  $P_0$  not only commutes with  $Q'$ , but also with all the elements of  $Q$ . Consequently  $G_0 := P_0 Q$  is an abelian group, and we have

$$[G : G_0] \leq (m+1)^c \leq (m+1)^r.$$

This completes the proof of the induction step, and with it that of Lemma 6.3.  $\square$

## 7. $\lambda$ -STABLE ACTIONS OF ABELIAN GROUPS

In this section all manifolds will be compact, possibly with boundary, and non necessarily connected. If  $X$  is a manifold we call the dimension of  $X$  (denoted by  $\dim X$ ) the maximum of the dimensions of the connected components of  $X$ .

### 7.1. Preliminaries.

**Lemma 7.1.** *Suppose that  $m, k$  are non negative integers. If  $X$  is a smooth manifold of dimension  $m$ ,  $X_1 \subset X_2 \subset \cdots \subset X_r \subseteq X$  are strict inclusions of neat<sup>4</sup> submanifolds, and each  $X_i$  has at most  $k$  connected components, then*

$$r \leq \binom{m+k+1}{m+1}.$$

*Proof.* Let  $X_1 \subset X_2 \subset \cdots \subset X_r \subseteq X$  be as in the statement of the lemma. For any  $i$  let  $d(X_i) = (d_m(X_i), \dots, d_0(X_i))$ , where  $d_j(X_i)$  denotes the number of connected components of  $X_i$  of dimension  $j$ . Each  $X_i$  has at most  $k$  connected components, so  $d(X_i)$  belongs to

$$\mathcal{D} = \{(d_m, \dots, d_1, d_0) \in \mathbb{Z}_{\geq 0}^{m+1} \mid \sum d_j \leq k\}.$$

Consider the lexicographic order on  $\mathcal{D}$ . We claim that for each  $i$  we have  $d(X_i) > d(X_{i-1})$ . To prove the claim, let us denote by  $X_{i-1,1}, \dots, X_{i-1,r}$  (resp.  $X_{i,1}, \dots, X_{i,s}$ ) the connected components of  $X_{i-1}$  (resp.  $X_i$ ), labelled in such a way that  $\dim X_{i-1,j-1} \geq \dim X_{i-1,j}$  and  $\dim X_{i,j-1} \geq \dim X_{i,j}$  for each  $j$ . Since  $X_{i-1} \subset X_i$ , there exists a map  $f : \{1, \dots, r\} \rightarrow$

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<sup>4</sup>See §1.4 in [18].



$\{1, \dots, s\}$  such that  $X_{i-1,j} \subseteq X_{i,f(j)}$ , which implies that  $\dim X_{i-1,j} \leq \dim X_{i,f(j)}$ . Let  $J$  be the set of indices  $j$  such that  $\dim X_{i-1,j} < \dim X_{i,f(j)}$ . We distinguish two cases.

If  $J = \emptyset$ , so that  $\dim X_{i-1,j} = \dim X_{i,f(j)}$  for each  $j$ , then  $X_{i-1} \neq X_i$  implies that  $X_i = X_{i-1} \sqcup X'_i$  for some nonempty and possibly disconnected  $X'_i \subset X$ , because by assumption  $X_{i-1}$  is a neat submanifold of  $X$ . This implies that  $d_\delta(X_i) \geq d_\delta(X_{i-1})$  for each  $\delta$ , and the inequality is strict for at least one  $\delta$ . Hence  $d(X_i) > d(X_{i-1})$ .

Now suppose that  $J \neq \emptyset$ . Let  $l = \dim X_{i,\min f(J)}$ . If  $l + 1 \leq \delta \leq m$  then any  $\delta$ -dimensional connected component of  $X_{i-1}$  is also a connected component of  $X_i$ , so  $d_\delta(X_i) \geq d_\delta(X_{i-1})$  (this is not necessarily an equality, since there might be some  $\delta$ -dimensional connected component of  $X_i$  which does not contain any connected component of  $X_{i-1}$ ), whereas  $d_l(X_i) > d_l(X_{i-1})$ . This implies again that  $d(X_i) > d(X_{i-1})$ , so the proof of the claim is complete.

The claim implies that  $r \leq |\mathcal{D}|$ , and an easy computation gives

$$|\mathcal{D}| = \binom{m}{m} + \binom{m+1}{m} + \dots + \binom{m+k}{m} = \binom{m+k+1}{m+1},$$

so the proof of the lemma is complete.  $\square$

The following result of Mann and Su (see [22, Theorem 2.5]) has already been mentioned in an earlier section: for any compact manifold  $X$  there exists some integer

$$\mu(X) \in \mathbb{Z}$$

with the property that for any prime  $p$  and any elementary  $p$ -group  $\mathbb{Z}_p^r$  admitting an effective action on  $X$  we have  $r \leq \mu(X)$ . This implies that any finite abelian  $p$ -group acting effectively on  $X$  is isomorphic to  $\mathbb{Z}_{p^{e_1}} \oplus \dots \oplus \mathbb{Z}_{p^{e_r}}$ , where  $r \leq \mu(X)$  and  $e_1, \dots, e_r$  are natural numbers.

**Lemma 7.2.** *Let  $X$  be a compact manifold and let  $n$  be a natural number. Any finite abelian  $p$ -group  $\Gamma$  acting effectively on  $X$  has a subgroup  $G \subset \Gamma$  of index  $[\Gamma : G] \leq p^{n\mu(X)}$  which is contained in each subgroup  $\Gamma' \subset \Gamma$  of index  $[\Gamma : \Gamma'] \leq p^n$ .*

*Proof.* We may assume that  $\Gamma \simeq \prod_{j=1}^r \langle \gamma_j \rangle$ , where  $r \leq \mu(X)$  and  $\gamma_1, \dots, \gamma_r \in \Gamma$ . We claim that  $G := \langle \gamma_1^{p^n}, \dots, \gamma_r^{p^n} \rangle$  has the desired property. Indeed, if  $\Gamma' \subset \Gamma$  is a subgroup satisfying  $[\Gamma : \Gamma'] \leq p^n$  then the exponent of  $\Gamma/\Gamma'$  divides  $p^n$ , and this implies that  $\gamma_j^{p^n} \in \Gamma'$  for each  $j$ , so  $G \subseteq \Gamma'$ . Clearly,  $[\Gamma : G] \leq p^{nr} \leq p^{n\mu(X)}$ .  $\square$

**Lemma 7.3.** *Let  $X$  be a compact smooth manifold and let  $p$  be any prime number. There exists a natural number  $C_{p,\chi}$  with the following property. For any finite  $p$ -group  $\Gamma$  acting smoothly on  $X$  there exists a subgroup  $\Gamma_\chi \subseteq \Gamma$  of index at most  $C_{p,\chi}$  satisfying*

- (1)  $[\Gamma : \Gamma_\chi] \leq C_{p,\chi}$ ;
- (2) for any subgroup  $\Gamma_0 \subseteq \Gamma_\chi$  we have  $\chi(X^{\Gamma_0}) = \chi(X)$ .

Furthermore, there exists some  $P_\chi$  such that if  $p \geq P_\chi$  then  $C_{p,\chi}$  can be taken to be 1.

*Proof.* Let  $n$  be the smallest integer such that  $p^{n+1} > 2 \sum_j b_j(X; \mathbb{F}_p)$ . We have

$$|\chi(X) + ap^{n+1}| > \sum_j b_j(X; \mathbb{F}_p) \quad \text{for any nonzero integer } a.$$

We are going to prove that  $C_{p,\chi} := p^{n\mu(X)}$  does the job. Let  $\Gamma$  be a  $p$ -group acting on  $X$ . Let  $\Gamma_{\text{tr}} \subseteq \Gamma$  be the kernel of the morphism  $\Gamma \rightarrow \text{Diff}(X)$  given by the action. For the purposes of proving the lemma we may replace  $\Gamma$  by  $\Gamma/\Gamma_{\text{tr}}$  and hence assume that  $\Gamma$  acts effectively on  $X$ .

Take, using Lemma 7.2, a subgroup  $\Gamma_\chi \subseteq \Gamma$  of index  $[\Gamma : \Gamma_\chi] \leq p^{n\mu(X)}$  such that for any subgroup  $\Gamma' \subseteq \Gamma$  of index  $[\Gamma : \Gamma'] \leq p^n$  we have  $\Gamma_\chi \subseteq \Gamma'$ . We now prove that any subgroup  $\Gamma_0 \subseteq \Gamma_\chi$  satisfies  $\chi(X^{\Gamma_0}) = \chi(X)$ . Consider a  $\Gamma$ -good triangulation  $(\mathcal{C}, \phi)$  of  $X$ . We have  $|\mathcal{C}|^{\Gamma_0} = |\mathcal{C}^{\Gamma_0}|$ , so

$$(5) \quad \chi(X) - \chi(X^{\Gamma_0}) = \chi(\mathcal{C}) - \chi(\mathcal{C}^{\Gamma_0}) = \sum_{j \geq 0} (-1)^j \#\{\sigma \in \mathcal{C} \setminus \mathcal{C}^{\Gamma_0} \mid \dim \sigma = j\}.$$

If  $\sigma \in \mathcal{C} \setminus \mathcal{C}^{\Gamma_0}$  then the stabilizer  $\Gamma_\sigma = \{\gamma \in \Gamma \mid \gamma \cdot \sigma = \sigma\}$  does not contain  $\Gamma_0$ . This implies that  $[\Gamma : \Gamma_\sigma] \geq p^{n+1}$ , for otherwise  $\Gamma_\sigma$  would contain  $\Gamma_\chi$  and hence also  $\Gamma_0$ . Consequently, the cardinal of the orbit  $\Gamma \cdot \sigma$  is divisible by  $p^{n+1}$ . Repeating this argument for all  $\sigma \in \mathcal{C} \setminus \mathcal{C}^{\Gamma_0}$  and using (5), we conclude that  $\chi(X) - \chi(X^{\Gamma_0})$  is divisible by  $p^{n+1}$ .

Now, we have  $|\chi(X^{\Gamma_0})| \leq \sum_j b_j(X^{\Gamma_0}; \mathbb{F}_p) \leq \sum_j b_j(X; \mathbb{F}_p)$  (the first inequality is obvious, and the second one follows from Lemma 2.3). By our choice of  $n$ , the congruence  $\chi(X^{\Gamma_0}) \equiv \chi(X) \pmod{p^{n+1}}$  and the inequality  $|\chi(X^{\Gamma_0})| \leq \sum_j b_j(X; \mathbb{F}_p)$  imply that  $\chi(X^{\Gamma_0}) = \chi(X)$ .

We now prove the last statement. Since  $X$  is compact, its cohomology is finitely generated, so in particular the torsion of its integral cohomology is bounded. Hence there exists some  $p_0$  such that if  $p \geq p_0$  then  $b_j(X; \mathbb{F}_p) = b_j(X)$  for every  $j$ . Define  $P_\chi = \max\{p_0, 2 \sum_j b_j(X) + 1\}$ . If  $p \geq P_\chi$  then the number  $n$  defined at the beginning of the proof is equal to 0, so  $C_{p,\chi}$  can be taken to be 1.  $\square$

**7.2.  $\lambda$ -stable actions: abelian  $p$ -groups.** Let  $p$  be a prime and let  $\Gamma$  be a finite abelian  $p$ -group acting smoothly on a smooth compact manifold  $X$ . Recall that for any  $x \in X^\Gamma$  the space  $T_x X / T_x^\Gamma X$  (which is the fiber over  $x$  of the normal bundle of the inclusion of  $X^\Gamma$  in  $X$ ) carries a linear action of  $\Gamma$ , induced by the derivative at  $x$  of the action on  $X$ , and depending up to isomorphism only on the connected component of  $X^\Gamma$  to which  $x$  belongs.

Let  $\lambda$  be a natural number. We say that the action of  $\Gamma$  on  $X$  is  **$\lambda$ -stable** if:

- (1)  $\chi(X^{\Gamma_0}) = \chi(X)$  for any subgroup  $\Gamma_0 \subseteq \Gamma$ ;
- (2) for any  $x \in X^\Gamma$  and any character  $\theta: \Gamma \rightarrow \mathbb{C}^*$  occurring in the  $\Gamma$ -module  $T_x X / T_x^\Gamma X$  we have

$$[\Gamma : \text{Ker } \theta] > \lambda.$$

Note that if  $\Gamma$  acts trivially on  $X$  then the action is  $\lambda$ -stable for any  $\lambda$ .

When the manifold  $X$  and the action of  $\Gamma$  on  $X$  are clear from the context, we will sometimes abusively say that  $\Gamma$  is  $\lambda$ -stable. For example, if a group  $G$  acts on  $X$  we will say that an abelian  $p$ -subgroup  $\Gamma \subseteq G$  is  $\lambda$ -stable if the restriction of the action of  $G$  to  $\Gamma$  is  $\lambda$ -stable.

**Lemma 7.4.** *Let  $\Gamma$  be a finite abelian  $p$ -group acting smoothly on  $X$  so that for any subgroup  $\Gamma' \subseteq \Gamma$  we have  $\chi(X^{\Gamma'}) = \chi(X)$ . If  $p > \lambda$  then  $\Gamma$  is  $\lambda$ -stable. If  $p \leq \lambda$  then there exists a  $\lambda$ -stable subgroup  $\Gamma_{\text{st}} \subseteq \Gamma$  satisfying*

$$[\Gamma : \Gamma_{\text{st}}] \leq \lambda^e, \quad e = \binom{m+k+1}{m+1},$$

where  $m = \dim X$  and  $k = \sum_j b_j(X; \mathbb{F}_p)$ .

*Proof.* Suppose that  $p$  is a prime number satisfying  $p > \lambda$ , and that  $\Gamma$  is a finite abelian  $p$ -group satisfying the properties in the statement of the lemma. Then  $\Gamma$  is  $\lambda$ -stable, because for any  $x \in X^\Gamma$  and any character  $\theta : \Gamma \rightarrow \mathbb{C}^*$  occurring in  $T_x X / T_x X^\Gamma$  the subgroup  $\text{Ker } \theta \subset \Gamma$ , being a strict subgroup (by (1) in Lemma 2.1), satisfies  $[\Gamma : \text{Ker } \theta] \geq p > \lambda$ .

Now suppose that  $p$  is a prime satisfying  $p \leq \lambda$  and that  $\Gamma$  is a finite abelian  $p$ -group satisfying the properties in the statement of the lemma. Let also  $m, k, e$  be as in the statement. We are going to prove that there exists some  $\lambda$ -stable subgroup  $\Gamma_{\text{st}} \subseteq \Gamma$  satisfying  $[\Gamma : \Gamma_{\text{st}}] \leq \lambda^e$ .

Let  $\Gamma_0 = \Gamma$ . If  $\Gamma_0$  is  $\lambda$ -stable, we define  $\Gamma_{\text{st}} := \Gamma_0$  and we are done. If  $\Gamma_0$  is not  $\lambda$ -stable, then there exists some  $x \in X^{\Gamma_0}$  and a character  $\theta : \Gamma_0 \rightarrow \mathbb{C}^*$  occurring in the  $\Gamma_0$ -module  $T_x X / T_x X^{\Gamma_0}$  such that  $[\Gamma_0 : \text{Ker } \theta] \leq \lambda$ . Choose one such  $x$  and  $\theta$  and define  $\Gamma_1 := \text{Ker } \theta$ . Then clearly  $[\Gamma_0 : \Gamma_1] \leq \lambda$  and, by (1) in Lemma 2.1,  $X^{\Gamma_0} \subset X^{\Gamma_1}$ . If  $\Gamma_1$  is  $\lambda$ -stable, then we define  $\Gamma_{\text{st}} := \Gamma_1$  and we stop, otherwise we repeat the same procedure with  $\Gamma_0$  replaced by  $\Gamma_1$  and define a subgroup  $\Gamma_2 \subset \Gamma_1$  satisfying  $[\Gamma_1 : \Gamma_2] \leq \lambda$  and  $X^{\Gamma_1} \subset X^{\Gamma_2}$ . And so on. Each time we repeat this procedure, we go from one group  $\Gamma_i$  to a subgroup  $\Gamma_{i+1}$  satisfying  $[\Gamma_i : \Gamma_{i+1}] \leq \lambda$  and  $X^{\Gamma_i} \subset X^{\Gamma_{i+1}}$ .

Suppose that we have been able to repeat the previous procedure  $e$  steps, so that we have a decreasing sequence of subgroups  $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \dots \supset \Gamma_e$  giving strict inclusions

$$X^{\Gamma_0} \subset X^{\Gamma_1} \subset \dots \subset X^{\Gamma_e} \subset X.$$

For each  $j$  the manifold  $X^{\Gamma_j}$  is a neat submanifold of  $X$  (by (1) in Lemma 2.1) and the number of connected components of  $X^{\Gamma_j}$  satisfies (by Lemma 2.3)

$$|\pi_0(X^{\Gamma_j})| = b_0(X^{\Gamma_j}; \mathbb{F}_p) \leq \sum_j b_j(X^{\Gamma_j}; \mathbb{F}_p) \leq \sum_j b_j(X; \mathbb{F}_p) = k.$$

So our assumption leads to a contradiction with Lemma 7.1. It follows that the previous procedure must stop before reaching the  $e$ -th step, so its outcome is a sequence of subgroups  $\Gamma = \Gamma_0 \supset \dots \supset \Gamma_f$  satisfying  $[\Gamma_i : \Gamma_{i+1}] \leq \lambda$ ,  $f < e$ , and  $\Gamma_{\text{st}} := \Gamma_f$  is  $\lambda$ -stable. We also have  $[\Gamma : \Gamma_{\text{st}}] \leq \lambda^f \leq \lambda^e$ , so the proof of the lemma is complete.  $\square$

**7.3. Fixed point sets and inclusions of groups.** Let  $X$  be a compact manifold. If  $A, B$  are submanifolds of  $X$ , we will write

$$A \preceq B$$

whenever  $A \subseteq B$  and each connected component of  $A$  is a connected component of  $B$ . Let  $p$  be a prime.

**Lemma 7.5.** *Let  $\lambda$  be a natural number. Let  $\Gamma$  be a finite abelian  $p$ -group acting smoothly on a compact manifold  $X$  in a  $\lambda$ -stable way. If a subgroup  $\Gamma_0 \subseteq \Gamma$  satisfies  $[\Gamma : \Gamma_0] \leq \lambda$  then  $X^\Gamma \preceq X^{\Gamma_0}$ .*

*Proof.* We clearly have  $X^\Gamma \subseteq X^{\Gamma_0}$ , so it suffices to prove that for each  $x \in X^\Gamma$  we have  $\dim_x X^\Gamma = \dim_x X^{\Gamma_0}$ . If this is not the case for some  $x \in X^\Gamma$  then, by (3) in Lemma 2.1, there exist an irreducible  $\Gamma$ -submodule of  $T_x X / T_x X^\Gamma$  on which the action of  $\Gamma_0$  is trivial. Let  $\theta: \Gamma \rightarrow \mathbb{C}^*$  be the character associated to this submodule. Then  $\Gamma_0 \subseteq \text{Ker } \theta$ , which implies that  $[\Gamma : \text{Ker } \theta] \leq \lambda$ , contradicting the hypothesis that  $\Gamma$  is  $\lambda$ -stable.  $\square$

**Lemma 7.6.** *Suppose that  $\lambda \geq (\dim X)(\sum_j b_j(X; \mathbb{F}_p))$ , and let  $\Gamma$  be a finite abelian  $p$ -group acting on  $X$  in a  $\lambda$ -stable way. There exists an element  $\gamma \in \Gamma$  such that  $X^\Gamma \preceq X^\gamma$ .*

*Proof.* Let  $\nu(\Gamma)$  be the collection of subgroups of  $\Gamma$  of the form  $\text{Ker } \theta$ , where  $\theta: \Gamma \rightarrow \mathbb{C}^*$  runs over the set of characters appearing in the action of  $\Gamma$  on the fibers of the normal bundle of the inclusion of  $X^\Gamma$  in  $X$ . Since  $\Gamma$  is finite, its representations are rigid, so the irreducible representations in the action of  $\Gamma$  on the normal fibers of the inclusion  $X^\Gamma \hookrightarrow X$  are locally constant on  $X^\Gamma$ . For each  $x \in X^\Gamma$  the representation of  $\Gamma$  on  $T_x X / T_x X^\Gamma$  splits as the sum of at most  $\dim X$  different irreducible representations. Consequently,  $\nu(\Gamma)$  has at most  $\dim X |\pi_0(X^\Gamma)|$  elements. By Lemma 2.3,  $|\pi_0(X^\Gamma)| \leq \sum_j b_j(X; \mathbb{F}_p)$ . Since  $\Gamma$  is  $\lambda$ -stable, we have  $|\Gamma'| < \lambda^{-1} |\Gamma|$  for each  $\Gamma' \in \nu(\Gamma)$ , so

$$\left| \bigcup_{\Gamma' \in \nu(\Gamma)} \Gamma' \right| \leq \lambda^{-1} |\Gamma| |\nu(\Gamma)| \leq \lambda^{-1} |\Gamma| \dim X \left( \sum_j b_j(X; \mathbb{F}_p) \right) < |\Gamma|.$$

Consequently, there exists at least one element  $\gamma \in \Gamma$  not contained in  $\bigcup_{\Gamma' \in \nu(\Gamma)} \Gamma'$ . By Lemma 2.1 we have  $X^\Gamma \preceq X^\gamma$ .  $\square$

**7.4.  $\lambda$ -stable actions: arbitrary abelian groups.** Let  $X$  be a compact manifold and let  $\Gamma$  be a finite abelian group acting on  $X$ . For any prime  $p$  we denote by  $\Gamma_p$  the  $p$ -part of  $\Gamma$ . We say that the action of  $\Gamma$  on  $X$  is  $\lambda$ -stable if and only if for any prime  $p$  the restriction of the action to  $\Gamma_p$  is  $\lambda$ -stable (recall that any action of the trivial group is  $\lambda$ -stable). As for  $p$ -groups, when the manifold  $X$  and the action are clear from the context, we will sometimes say that  $\Gamma$  is  $\lambda$ -stable (this will be often the case when talking about subgroups of a group acting on  $X$ ).

**Theorem 7.7.** *Let  $\lambda$  be a natural number. There exists a constant  $C_\lambda$ , depending only on  $X$  and  $\lambda$ , such that any finite abelian group  $\Gamma$  acting on  $X$  has a  $\lambda$ -stable subgroup of index at most  $C_\lambda$ .*

*Proof.* Let  $P_\chi$  be the number defined in Lemma 7.3. Define

$$C_\lambda := \left( \prod_{p \leq P_\chi} C_{p,\chi} \right) \left( \prod_{p \leq \lambda} \lambda^e \right),$$

where in both products  $p$  runs over the set of primes satisfying the inequality and

$$e = \binom{m + K + 1}{m + 1}, \quad m = \dim X, \quad K = \sum_j \max\{b_j(X; \mathbb{F}_p) \mid p \text{ prime}\}.$$

The theorem follows from combining Lemmas 7.3 and Lemma 7.4 applied to each of the factors of  $\Gamma \simeq \prod_{p|d} \Gamma_p$ , where  $d = |\Gamma|$ .  $\square$

**7.5.  $\lambda$ -stable actions on manifolds without odd cohomology.** In this section  $X$  denotes a manifold without odd cohomology. Let  $p$  be any prime number. Applying cohomology to the exact sequence of locally constant sheaves on  $X$

$$0 \rightarrow \underline{\mathbb{Z}} \xrightarrow{p} \underline{\mathbb{Z}} \rightarrow \underline{\mathbb{F}_p} \rightarrow 0$$

and using the fact that  $X$  has no odd cohomology we obtain

$$(6) \quad b_j(X; \mathbb{F}_p) = b_j(X) \quad \text{for any } j \quad \implies \quad \chi(X) = \sum_j b_j(X) = \sum_j b_j(X; \mathbb{F}_p).$$

**Lemma 7.8.** *Let  $p$  be any prime number. Suppose that a finite abelian  $p$ -group  $\Gamma$  acts on  $X$  and that there is a subgroup  $\Gamma' \subseteq \Gamma$  such that  $X^\Gamma \preceq X^{\Gamma'}$  and  $\chi(X^\Gamma) = \chi(X^{\Gamma'}) = \chi(X)$ . Then  $X^\Gamma = X^{\Gamma'}$ .*

*Proof.* By Lemma 2.3 we have  $\sum_j b_j(X^\Gamma; \mathbb{F}_p) \leq \sum_j b_j(X; \mathbb{F}_p)$  so, using (6),

$$\chi(X^\Gamma) \leq \sum_j b_j(X^\Gamma; \mathbb{F}_p) \leq \sum_j b_j(X; \mathbb{F}_p) = \chi(X).$$

Since  $\chi(X^\Gamma) = \chi(X)$  we have  $\sum_j b_j(X^\Gamma; \mathbb{F}_p) = \sum_j b_j(X; \mathbb{F}_p)$ . Applying the same arguments to  $\Gamma'$  we conclude that  $\sum_j b_j(X^\Gamma; \mathbb{F}_p) = \sum_j b_j(X^{\Gamma'}; \mathbb{F}_p)$ . Combining this with  $X^\Gamma \preceq X^{\Gamma'}$  we deduce  $X^\Gamma = X^{\Gamma'}$ .  $\square$

**Lemma 7.9.** *Let  $\lambda_\chi = \chi(X) \dim X$ . If  $\Gamma$  is a finite abelian  $p$ -group acting on  $X$  in a  $\lambda_\chi$ -stable way, then there exists some  $\gamma \in X$  such that  $X^\Gamma = X^\gamma$ .*

*Proof.* This follows from combining Lemma 7.6, equality (6), and Lemma 7.8.  $\square$

## 8. PROOF OF THEOREM 1.3

Let  $X$  be a compact manifold without odd cohomology. Let  $A$  be a finite abelian group acting smoothly on  $X$ .

Let  $b_j = b_j(X)$  and let  $b = \sum_j b_j^2$ . We claim that there exists a subgroup  $G \subseteq A$  whose action on the cohomology  $H^*(X; \mathbb{Z})$  is trivial and which satisfies

$$[A : G] \leq 3^b.$$

To prove the claim, recall that a well known lemma of Minkowski states that for any  $n$  and any finite group  $H \subseteq \mathrm{GL}(n, \mathbb{Z})$  the restriction of the quotient map

$$q_n : \mathrm{GL}(n, \mathbb{Z}) \rightarrow \mathrm{GL}(n, \mathbb{F}_3)$$

to  $H$  is injective (see e.g. [24, 37]; the proof is easy: it suffices to check that for any nonzero  $M \in \mathrm{Mat}_{n \times n}(\mathbb{Z})$  and nonzero integer  $k$  the matrix  $(\mathrm{Id}_n + 3M)^k$  is different from the identity, see e.g. [11, V.3.4]). Choosing a homogeneous basis of  $H^*(X; \mathbb{Z})$  the action of  $A$  on the cohomology can be encoded in a morphism of groups

$$\phi : A \rightarrow \prod_j \mathrm{GL}(b_j, \mathbb{Z}).$$

Then

$$G := \mathrm{Ker}(q \circ \phi), \quad q = (q_{b_0}, \dots, q_{b_n}), \quad n = \dim X$$

has the required property, because  $|\prod_j \mathrm{GL}(b_j, \mathbb{F}_3)| \leq 3^b$ .

Let  $\lambda_X = \chi(X) \dim X$ . By Theorem 7.7 there exists a subgroup  $\Gamma \subseteq G$  whose action on  $X$  is  $\lambda_X$ -stable and which satisfies  $[G : \Gamma] \leq C_{\lambda_X}$ , where  $C_{\lambda_X}$  depends on  $\lambda_X$  and  $X$ , but not on the group  $G$ .

There is an isomorphism  $\Gamma \simeq \Gamma_{p_1} \times \dots \times \Gamma_{p_k}$ , where  $p_1, \dots, p_k$  are the prime divisors of  $|\Gamma|$ . Since the action of  $\Gamma$  is  $\lambda_X$ -stable so is, by definition, its restriction to each  $\Gamma_{p_i}$ , so by Lemma 7.9 there exists, for each  $i$ , an element  $\gamma_i \in \Gamma_{p_i}$  such that  $X^{\gamma_i} = X^{\Gamma_{p_i}}$ . Let  $\gamma = \gamma_1 \dots \gamma_k$ . Then  $X^\Gamma = \bigcap_i X^{\Gamma_i} \subseteq X^\gamma$ . By the Chinese remainder theorem and the fact that the elements  $\gamma_1, \dots, \gamma_k$  commute, for each  $i$  there exists some  $e$  such that  $\gamma^e = \gamma_i$ . Hence  $X^\gamma \subseteq X^{\gamma^e} = X^{\gamma_i} = X^{\Gamma_{p_i}}$ . Taking the intersection over all  $i$  we get  $X^\gamma \subseteq \bigcap_i X^{\Gamma_i} = X^\Gamma$ . Combining the two inclusions we have  $X^\gamma = X^\Gamma$ .

Since  $\gamma \in G$ , the action of  $\gamma$  on  $X$  induces the trivial action on  $H^*(X; \mathbb{Z})$ , so in particular it preserves the connected components of  $X$ . Let  $Y \subseteq X$  be any connected component. Applying Lefschetz's formula [9, Exercise 6.17.3] to the action of  $\gamma$  on  $Y$  we conclude that  $\chi(Y^\gamma) = \chi(Y)$ . Since  $Y^\gamma = Y^\Gamma$ , it follows that  $A_0 := \Gamma$  has the desired properties. Finally,

$$[A : A_0] \leq 3^b C_{\lambda_X}.$$

## 9. PROOFS OF THEOREMS 1.5 AND 1.6

**9.1. Proof of Theorem 1.5.** The following well known fact immediately proves the theorem in the cases  $n = 1, 2$ .

**Lemma 9.1.** *Let  $n$  be either 1 or 2, and let  $X$  be the  $n$ -dimensional disk. The fixed point locus of any smooth action of a finite group on  $X$  is contractible.*

Let now  $n \geq 3$  be a natural number and let  $X$  be the  $n$ -dimensional disk. Let  $p$  be a prime and suppose that a finite abelian  $p$ -group  $A$  acts on  $X$ . Smith theory implies that the fixed point set  $X^A$  is  $\mathbb{F}_p$ -acyclic (see the proof of Theorem 5.1). In particular,  $X^A$  is nonempty and connected.

**Lemma 9.2.** *Let  $x \in X^A$  be any point, and let  $\theta_1, \dots, \theta_r$  ( $\theta_j : A \rightarrow \text{GL}(W_j)$ ) be the different (real) irreducible representations of  $A$  appearing in  $T_x X / T_x X^A$ . Let  $l = \dim X^A$ .*

- (1) *If  $p = 2$  then  $r \leq n - l$ . If  $p$  is odd then  $n - l$  is even and  $r \leq (n - l)/2$ .*
- (2) *There exists some  $\gamma \in A$  and a subgroup  $A' \subseteq A$  such that  $X^\gamma = X^{A'}$  and  $[A : A']$  divides  $p^{\lceil r/p \rceil}$ .*

*Proof.* (1) is clear. To prove (2), define  $A_j := \text{Ker } \theta_j$ , let  $e_j = \log_p [A : A_j]$ , and consider the function  $I : A \rightarrow \mathbb{Z}$  defined as  $I(\gamma) = \sum_{j|\gamma \in A_j} e_j$ . Since each  $\theta_j$  is nontrivial (e.g. by (2) in Lemma 2.1) we have  $e_j \geq 1$  for every  $j$ . Now we estimate

$$\sum_{\gamma \in A} I(\gamma) = \sum_{\gamma \in A} \sum_{j|\gamma \in A_j} e_j = \sum_{j=1}^r |A_j| e_j = \sum_{j=1}^r \frac{|A|}{p^{e_j}} e_j \leq \sum_{j=1}^r \frac{|A|}{p} = \frac{r}{p} |A|,$$

where the inequality follows from the fact that the function  $\mathbb{N} \ni n \mapsto n/q^n$  is non increasing for any integer  $q \geq 2$ . Hence the average value of  $I$  is not bigger than  $r/p$ , so there exists some  $\gamma \in A$ , which we fix for the rest of the argument, such that  $I(\gamma) \leq \lceil r/p \rceil$ . Let  $A' := \bigcap_{j|\gamma \in A_j} A_j$ .

We claim that  $X^\gamma = X^{A'}$ . Since  $\gamma \in A'$ , the inclusion  $X^{A'} \subseteq X^\gamma$  is clear. To prove the reverse inclusion observe that, by Smith theory, both  $X^{A'}$  and  $X^\gamma$  are acyclic, hence connected, so it suffices to prove that  $T_x X^\gamma \subseteq T_x X^{A'}$ . Let  $T_x X / T_x X^A = V_1 \oplus \dots \oplus V_r$  be the decomposition in isotypical real representations of  $A$ , where  $V_j$  is isomorphic to the direct sum of a number of copies of  $W_j$ . Since we clearly have

$$T_x X^A \oplus \bigoplus_{j|\gamma \in A_j} V_j \subseteq T_x X^{A'},$$

it suffices to prove that

$$T_x X^\gamma \subseteq T_x X^A \oplus \bigoplus_{j|\gamma \in A_j} V_j.$$

The latter is equivalent to proving that  $T_x X^\gamma \cap V_i = \text{Ker}(\theta_i^V(\gamma) - \text{Id}) = \{0\}$  for every  $i$  such that  $\gamma \notin A_i$  (here  $\theta_i^V : A \rightarrow \text{GL}(V_i)$  is given by restricting the action of  $A$  on  $T_x X / T_x X^A$ ). Since  $V_i$  is isotypical and  $\gamma$  is central in  $A$ ,  $\text{Ker}(\theta_i^V(\gamma) - \text{Id})$  is either  $\{0\}$  or  $V_i$ . But  $\text{Ker}(\theta_i^V(\gamma) - \text{Id}) = V_i$  would imply  $\gamma \in \text{Ker } \theta_i$ , contradicting the choice of  $i$ . Hence  $\text{Ker}(\theta_i^V(\gamma) - \text{Id}) = \{0\}$  and the proof that  $X^\gamma = X^{A'}$  is complete.

To finish the proof of the lemma we observe that  $[A : A']$  divides  $\prod_{j|\gamma \in A_j} [A : A_j] = p^{I(\gamma)}$ . Since  $I(\gamma) \leq \lceil r/p \rceil$ , we deduce that  $[A : A']$  divides  $p^{\lceil r/p \rceil}$ .  $\square$

Now let  $A$  be a finite abelian group acting on  $X$ . For each prime  $p$  let  $A_p \subseteq A$  denote the  $p$ -part of  $A$ , so that  $A = \prod_p A_p$ . If for some  $p$  we have  $\dim X^{A_p} \leq 2$ , then the

classification of manifolds (with boundary) of dimension at most 2 implies that  $\dim X^{A_p}$  is a disk, because  $\dim X^{A_p}$  is  $\mathbb{F}_p$ -acyclic; so applying Lemma 9.1 to the action of  $A$  on  $X^{A_p}$  we deduce that  $\chi(X^A) = \chi((X^{A_p})^A) = 1$  and the proof of the theorem is complete.

Hence it suffices to consider the case when  $\dim X^{A_p} \geq 3$  for each  $p$ . Let  $k = [(n-3)/2]$ . Applying Lemma 9.2 for each prime  $p$  we deduce that there exists some subgroup  $A'_2 \subseteq A_2$  satisfying  $[A_2 : A'_2] \leq 2^k$  and an element  $\gamma_2 \in A'_2$  such that  $X^{\gamma_2} = X^{A'_2}$  and, for each odd prime  $p$ , there exists some subgroup  $A'_p \subseteq A_p$  satisfying  $[A_p : A'_p] \leq p^{[k/p]}$  and an element  $\gamma_p \in A'_p$  such that  $X^{\gamma_p} = X^{A'_p}$ . Let  $A' = \prod A'_p$  and let  $\gamma = \prod \gamma_p$ . Arguing as in Section 8 we prove that  $X^\gamma = X^{A'}$ . Clearly  $[A : A']$  divides  $f(k)$ , so statement (1) of Theorem 1.5 is proved. Statement (2) follows immediately from statement (1), because none of the odd prime divisors of  $f(k)$  is bigger than  $k$ .

**9.2. Proof of Theorem 1.6.** We follow a scheme similar to the proof of Theorem 1.5. Let  $m$  be a natural number, let  $p$  be a prime and let  $Y$  be a  $\mathbb{F}_p$ -homology  $2m$ -sphere. For any smooth action of  $\mathbb{Z}_p$  on  $Y$  the fixed point set is a  $\mathbb{F}_p$ -homology  $s$ -sphere [2, IV.4.3]. Furthermore, if  $p$  is odd, the difference  $2m - s$  is even [2, IV.4.4]. Hence we may apply the same inductive scheme as in Lemma 2.3 (or [2, IV.4.5]) and deduce the following.

**Lemma 9.3.** *For any odd prime  $p$ , any finite  $p$ -group  $A$ , and any action of  $A$  on a  $\mathbb{F}_p$ -homology even dimensional sphere the fixed point set is a  $\mathbb{F}_p$ -homology even dimensional sphere (in particular, the fixed point set is nonempty).*

The case  $p = 2$  works differently. Suppose that  $Y$  is a smooth  $n$ -dimensional manifold and that  $Y$  is a  $\mathbb{F}_2$ -homology  $n$ -sphere. Suppose that  $\mathbb{Z}_2$  acts smoothly on  $Y$ . Then  $Y^{\mathbb{Z}_2}$  is a  $\mathbb{F}_2$ -homology  $s$ -sphere [2, IV.4.3]. The fixed point set  $Y^{\mathbb{Z}_2}$  is also a smooth submanifold of  $Y$  and  $s$  coincides with the dimension of  $Y^{\mathbb{Z}_2}$  as a manifold. The condition that  $Y$  is a  $\mathbb{F}_2$ -homology sphere implies that  $Y$  is compact and orientable. One checks, using Lefschetz' formula [9, Exercise 6.17.3] and arguing in terms of volume forms, that  $n - s$  is even if and only if the action of the nontrivial element of  $\mathbb{Z}_2$  on  $Y$  is orientation preserving (this works more generally for continuous actions on finite Hausdorff spaces whose integral homology is finitely generated and whose  $\mathbb{F}_2$ -homology is isomorphic to  $H_*(S^n; \mathbb{F}_2)$ , by a theorem of Liao [21], see also [2, IV.4.7]).

**Lemma 9.4.** *For any finite 2-group  $A$  and any smooth action of  $A$  on a smooth  $\mathbb{F}_2$ -homology  $2m$ -sphere ( $m \in \mathbb{Z}_{\geq 0}$ ) there exists a subgroup  $A_0 \subseteq A$  whose index  $[A : A_0]$  divides  $2^{m+1}$  and whose fixed point set  $X^{A_0}$  is a smooth  $\mathbb{F}_2$ -homology even dimensional sphere (in particular,  $X^{A_0}$  is nonempty).*

*Proof.* We use ascending induction on  $|A|$ . The case  $|A| = 2$  being obvious, suppose that  $|A| > 2$  and that the lemma is true for 2-groups with less elements than  $A$ . Suppose that  $A$  acts smoothly on a compact smooth  $\mathbb{F}_2$ -homology  $2r$ -sphere  $Y$ . If the action of  $A$  is not effective, then it factors through a quotient of  $A$ , and applying the inductive hypothesis the lemma follows. So assume that the action of  $A$  on  $Y$  is effective. Let  $A' \subseteq A$  be the subgroup consisting of those elements whose action is orientation preserving. Then



$[A : A']$  divides 2. Let  $A'' \subseteq A'$  be a central subgroup isomorphic to  $\mathbb{Z}_2$ . Then  $Y^{A''}$  is a compact smooth  $\mathbb{F}_2$ -homology even dimensional sphere satisfying  $\dim Y^{A''} \leq 2r - 2$ . Furthermore,  $A'/A''$  acts smoothly on  $Y^{A''}$ . To finish the proof, apply the inductive hypothesis to this action.  $\square$

Let now  $X$  be a smooth homology  $2r$ -sphere and suppose that a finite abelian group  $A$  acts smoothly on  $X$ . For any prime  $p$  let  $A_p \subseteq A$  denote the  $p$ -part. By Lemma 9.3, for any odd prime  $p$  the fixed point set  $X^{A_p}$  is an even dimensional  $\mathbb{F}_p$ -homology sphere. By Lemma 9.4,  $A_2$  has a subgroup  $A_{2,0}$  whose index divides  $2^{r+1}$  and whose fixed point set is an even dimensional  $\mathbb{F}_2$ -homology sphere (in particular, it is nonempty). Replace  $A_2$  by  $A_{2,0}$  and define  $A_0 := \prod_p A_p$ , so that  $[A : A_0]$  divides  $2^{r+1}$ .

Suppose that for some prime  $p$  the fixed point set  $X^{A_p}$  is 0-dimensional. Then  $X^{A_p}$  consists of two points,  $A$  acts on  $X^{A_p}$ , and there is a subgroup  $A' \subseteq A$  whose index divides 2 and whose action on  $X^{A_p}$  is trivial. It follows that  $|X^{A'}| \geq 2$  and we are done.

Suppose now that for each prime  $p$  the fixed point set  $X^{A_p}$  is an even dimensional  $\mathbb{F}_p$ -homology sphere of dimension at least 2. In particular,  $X^{A_p}$  is nonempty and connected for every  $p$ , and so is  $X^{A'_p}$  for every subgroup  $A'_p \subseteq A_p$  (since  $X^{A'_p}$  is a  $\mathbb{F}_p$ -homology sphere and, given the inclusion  $X^{A_p} \subseteq X^{A'_p}$ , the dimension of  $X^{A'_p}$  is at least 2). This property allows to use the same arguments in the proof of Lemma 9.2 to prove the following.

**Lemma 9.5.** *Let  $p$  be any prime, let  $x \in X^{A_p}$  be any point, and let  $\theta_1, \dots, \theta_r$  be the different real irreducible representations of  $A_p$  appearing in  $T_x X / T_x X^{A_p}$ . Let  $2l = \dim X^{A_p}$ .*

- (1) *If  $p = 2$  then  $r \leq 2m - 2l$ . If  $p$  is odd then  $r \leq m - l$ .*
- (2) *There exists some  $\gamma \in A_p$  and a subgroup  $A'_p \subseteq A_p$  such that  $X^\gamma = X^{A'_p}$  and  $[A_p : A'_p]$  divides  $p^{\lceil r/p \rceil}$ .*

By Lemma 9.5 there exists some subgroup  $A'_2 \subseteq A_2$  satisfying  $[A_2 : A'_2] \leq 2^{2m-2}$  and an element  $\gamma_2 \in A'_2$  such that  $X^{\gamma_2} = X^{A'_2}$  and, for each odd prime  $p$ , there exists some subgroup  $A'_p \subseteq A_p$  satisfying  $[A_p : A'_p] \leq p^{\lceil m-1/p \rceil}$  and an element  $\gamma_p \in A'_p$  such that  $X^{\gamma_p} = X^{A'_p}$ . Let  $A' = \prod A'_p$  and let  $\gamma = \prod \gamma_p$ . As in Section 8 we have  $X^\gamma = X^{A'}$ . The index  $[A : A']$  divides  $2^{m+1} f(m-1)$ , so statement (1) of Theorem 1.6 is proved. Statement (2) follows immediately from statement (1), because none of the odd prime divisors of  $f(m-1)$  is bigger than  $m-1$ .

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